R. Review Materials

Contents

R1 Mathematical Formulas and Identities

R1.1 Finite and Infinite Sums of Numbers[1]

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \tag{R1.1}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \tag{R1.2}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}, \tag{R1.3}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \tag{R1.4}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}, \tag{R1.5}$$

where n is a positive integer.

Note: The series

$$\sum_{k=1}^{\infty} \frac{1}{k} \tag{R1.6}$$

does not converge.

$$\sum_{k=0}^{n-1} d^k = \frac{1-d^n}{1-d}, \tag{R1.7}$$

with d arbitrary integer and $|d| \neq \{1,\ 0\}$.

$$\sum_{k=0}^{\infty} d^k = \frac{1}{1-d}, \tag{R1.8}$$

with $0 < |d| < 1$.

[1]For test of the convergence of infinite sums, a recommended reading is *Table of Integrals, Series, and Products*, I.S. Gradshteyn and I.M. Ryzhik, ©2000, Academic Press.

R1.2 Power Series

Binomial Series:

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots + \binom{n}{n-1}xy^{n-1} + y^n, \tag{R1.9}$$

with n a positive integer.

Taylor Series:

If $f(x)$ is an arbitrarily differentiable function, then it can be expressed in the form

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots\ , \tag{R1.10}$$

where $f'(a) = \frac{df(x)}{dx}\Big|_{x=a}$, $f''(a) = \frac{d^2f(x)}{dx^2}\Big|_{x=a}$, and $f^{(n)}(a) = \frac{d^nf(x)}{dx^n}\Big|_{x=a}$.

Exponential Series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\ , \tag{R1.11}$$

$$a^x = 1 + x\log_e a + \frac{(x\log_e a)^2}{2!} + \frac{(x\log_e a)^3}{3!} + \cdots\ . \tag{R1.12}$$

Logarithmic Series:

$$\log_e(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots\ , (\text{Region of Convergence} : -1 < x < 1). \tag{R1.13}$$

Trigonometric Series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \tag{R1.14}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \tag{R1.15}$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots, \text{ (Region of Convergence} : x^2 < \tfrac{\pi^2}{4}), \tag{R1.16}$$

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} + \cdots, \text{ (Region of Convergence} : 0 < |x| < \pi). \tag{R1.17}$$

R1.3 Factorial

$$n! = n(n-1)(n-2)\cdots 2 \cdot 1, \quad \text{with } n \text{ a nonnegative integer}, \tag{R1.18}$$

$$0! = \Gamma(0+1) = 1. \tag{R1.19}$$

R1.4 Permutations and Combinations

The number of permutations S of n things taken k at a time, with n and k positive integers, is given by

$$S = \frac{n!}{(n-k)!}. \tag{R1.20}$$

The number of combinations S of n things taken k at a time, with n and k positive integers, is given by

$$S = \tbinom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{R1.21}$$

R1.5 Polynomial Factors and Products

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + y^{n-1}), \tag{R1.22}$$

with n a positive integer.

$$x^n + y^n = (x+y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots + y^{n-1}), \tag{R1.23}$$

with n a positive and odd integer.

$$\begin{aligned}\prod_{i=1}^{N}(x+\xi_i) &= (x+\xi_1)(x+\xi_2)\cdots(x+\xi_N) \\ &= \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_{N-1}x^{N-1} + \alpha_N x^N\end{aligned} \tag{R1.24}$$

where

$$\alpha_0 = \prod_{i=1}^{N}\xi_i, \quad \alpha_1 = \sum_{i=1}^{N}\frac{\alpha_0}{\xi_i}, \quad \alpha_2 = \sum_{\substack{i\neq j\\ i,j=1}}^{N}\frac{\alpha_0}{\xi_i\xi_j}, \quad \cdots,$$

$$\alpha_{N-1} = \xi_1 + \xi_2 + \xi_3 + \cdots + \xi_N, \quad \alpha_N = 1.$$

R1.6 Roots of Quadratic Equation

The roots x_1, x_2 of the quadratic equation

$$ax^2 + bx + c = 0,$$

with a, b, and c real numbers, are given by

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \tag{R1.25}$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{R1.26}$$

Note:

$$x_1 + x_2 = \frac{-b}{a}, \tag{R1.27}$$

$$x_1 x_2 = \frac{c}{a}. \tag{R1.28}$$

R1.7 Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta, \tag{R1.29}$$

with θ a real number.

R1.8 Trigonometric Functions and Formulas

$$\sin\theta = \tfrac{1}{2j}(e^{j\theta} - e^{-j\theta}), \tag{R1.30}$$

$$\cos\theta = \tfrac{1}{2}(e^{j\theta} + e^{-j\theta}), \tag{R1.31}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{(e^{j\theta} - e^{-j\theta})}{j(e^{j\theta} + e^{-j\theta})}, \tag{R1.32}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{j(e^{j\theta} + e^{-j\theta})}{(e^{j\alpha} - e^{-j\theta})}, \tag{R1.33}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{2j}{e^{j\theta} - e^{-j\theta}}, \tag{R1.34}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{2}{e^{j\theta} + e^{-j\theta}}, \tag{R1.35}$$

$$\sin\theta = \cos(\tfrac{\pi}{2} - \theta) = \sin(\pi - \theta), \tag{R1.36}$$

$$\cos\theta = \sin(\tfrac{\pi}{2} - \theta) = -\cos(\pi - \theta), \tag{R1.37}$$

$$\tan\theta = \cot(\tfrac{\pi}{2} - \theta) = -\tan(\pi - \theta), \tag{R1.38}$$

$$\sinh\theta = \tfrac{1}{2}(e^{\theta} - e^{-\theta}), \tag{R1.39}$$

$$\cosh\theta = \tfrac{1}{2}(e^{\theta} + e^{-\theta}), \tag{R1.40}$$

$$\tanh\theta = \frac{\sinh\theta}{\cosh\theta} = \frac{(e^{\theta} - e^{-\theta})}{(e^{\theta} + e^{-\theta})}, \tag{R1.41}$$

with θ a real number.

$$
\begin{aligned}
\sin(\theta_1 \pm \theta_2) &= \sin\theta_1 \cos\theta_2 \pm \cos\theta_1 \sin\theta_2, && \text{(R1.42)}\\
\cos(\theta_1 \pm \theta_2) &= \cos\theta_1 \cos\theta_2 \mp \sin\theta_1 \sin\theta_2, && \text{(R1.43)}\\
\sin^2\theta_1 - \sin^2\theta_2 &= \sin(\theta_1+\theta_2)\cdot\sin(\theta_1-\theta_2), && \text{(R1.44)}\\
\cos^2\theta_1 - \cos^2\theta_2 &= -\sin(\theta_1+\theta_2)\cdot\sin(\theta_1-\theta_2), && \text{(R1.45)}\\
\cos^2\theta_1 - \sin^2\theta_2 &= \cos(\theta_1+\theta_2)\cdot\cos(\theta_1-\theta_2), && \text{(R1.46)}\\
\cos^2\theta_1 + \sin^2\theta_2 &= 1 && \text{(R1.47)}\\
\sin\theta_1 \pm \sin\theta_2 &= 2\sin\Big(\frac{\theta_1\pm\theta_2}{2}\Big)\cdot\cos(\theta_1\mp\theta_2), && \text{(R1.48)}\\
\cos\theta_1 + \cos\theta_2 &= 2\cos\Big(\frac{\theta_1+\theta_2}{2}\Big)\cdot\cos\Big(\frac{\theta_1-\theta_2}{2}\Big), && \text{(R1.49)}\\
\cos\theta_1 - \cos\theta_2 &= -2\sin\Big(\frac{\theta_1+\theta_2}{2}\Big)\cdot\sin\Big(\frac{\theta_1-\theta_2}{2}\Big), && \text{(R1.50)}\\
\sin 2\theta &= 2\sin\theta\cos\theta, && \text{(R1.51)}\\
\cos 2\theta &= \cos^2\theta - \sin^2\theta, && \text{(R1.52)}\\
\sin 3\theta &= 3\sin\theta - 4\sin^3\theta, && \text{(R1.53)}\\
\cos 3\theta &= 4\cos^3\theta - 3\cos\theta. && \text{(R1.54)}
\end{aligned}
$$

with θ, θ_1, and θ_2 real numbers.

R1.9 Newton-Raphson Method: Finding a root of a polynomial equation

The Newton-Raphson method is a numerical technique to determine approximately the root of the equation $f(x) = 0$. The procedure starts from a initial guess of the root $x = x_1$. Then using the recurrence relation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 1, 2, \cdots,$$

where

$$f'(x_n) = \frac{df(x)}{dx}\Big|_{x=x_n},$$

the successive approximations $x_{n+1}, n \geq 1$, beginning with $n = 1$, can be found. The approximation is assumed to converge when the difference between x_{n+1} and x_n is below a prescribed small number, typically 10^{-6}.

The Newton-Raphson method converges fast to the actual root if the initial guess of the root is close to the actual root. However, there are three main drawbacks: (1) The method fails when $f'(x_n) = 0$, (2) The method does not always converge, and (3) The method may converge to a root different from that expected if the initial guess x_1 is far from the actual root.

Example R1.1. In this example, we would like to show how the Newton-Raphson method is used to find the root of $f(x) = x^3 - 3x^2 + x - 1 = 0$. Assume the numerical resolution required is 14 decimal digits.

We start with an initial guess of the root $x_1 = 2.5$:

$$\begin{aligned}
x_1 &= 2.5, \\
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 2.84210526315789, \\
x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = 2.77282691999216, \\
x_4 &= x_3 - \frac{f(x_3)}{f'(x_3)} = 2.76930129255045, \\
x_5 &= x_4 - \frac{f(x_4)}{f'(x_4)} = 2.76929235429601, \\
x_6 &= x_5 - \frac{f(x_5)}{f'(x_5)} = 2.76929235423863, \\
x_7 &= x_6 - \frac{f(x_6)}{f'(x_6)} = 2.76929235423863.
\end{aligned}$$

The recurrence process stops as $|x_7 - x_6| \leq 10^{-15}$. Hence $x = x_7$ is a root of $f(x)$.

R1.10 Hölder's Inequality and Cauchy-Schwartz's Inequality

The Hölder's inequality for integrals is given by

$$\left| \int_{a_0}^{a_1} f(x)g(x)\,dx \right| \leq \left(\int_{a_0}^{a_1} |f(x)|^p\,dx \right)^{1/p} \left(\int_{a_0}^{a_1} |g(x)|^q\,dx \right)^{1/q}, \qquad \text{(R1.55)}$$

where

$$\frac{1}{p} + \frac{1}{q} = 1.$$

The equality holds when

$$f(x) = kg(x)^{p-1}, \text{with } k \text{ any constant.}$$

If $p = q = 2$, the inequality becomes Schwartz's inequality

$$\left| \int_{a_0}^{a_1} f(x)g(x)\, dx \right| \le \left(\int_{a_0}^{a_1} |f(x)|^2\, dx \right)^{1/2} \left(\int_{a_0}^{a_1} |g(x)|^2\, dx \right)^{1/2}. \qquad \text{(R1.56)}$$

The equality holds when

$$f(x) = kg(x), \text{with } k \text{ any constant.}$$

The Hölder's inequality for sums is given by

$$\left| \sum_{i=1}^{N} x_i y_i \right| \le \left(\sum_{i=1}^{N} |x_i|^p \right)^{1/p} \left(\sum_{i=1}^{N} |y_i|^q \right)^{1/q}, \qquad \text{(R1.57)}$$

where

$$\frac{1}{p} + \frac{1}{q} = 1.$$

The equality holds when

$$y_i = kx_i^{p-1}, \text{with } k \text{ any constant.}$$

If $p = q = 2$, the inequality becomes Cauchy's inequality

$$\left| \sum_{i=1}^{N} x_i y_i \right| \le \left(\sum_{i=1}^{N} |x_i|^2 \right)^{1/2} \left(\sum_{i=1}^{N} |y_i|^2 \right)^{1/2}, \qquad \text{(R1.58)}$$

The equality holds when

$$y_i = kx_i, \text{with } k \text{ any constant.}$$

R2 Useful Functions

1. *rect* function
$$\text{rect}(x) = \begin{cases} 1, & |x| < \frac{1}{2} \\ 0, & |x| > \frac{1}{2} \end{cases}.$$

2. *sinc* function
$$\text{sinc}(x) = \frac{\sin \pi x}{\pi x}.$$

3. *signum* function
$$\text{sgn}(x) = \begin{cases} +1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}.$$

4. *ceiling* function rounds the input x towards the closest integer larger than or equal to x and is denoted as $\lceil x \rceil$.

   For example, $\lceil 3.2 \rceil = \lceil 3.8 \rceil = 4$ and $\lceil -3.2 \rceil = \lceil -3.9 \rceil = -3$.

5. *floor* function rounds the input x towards the closest integer less than or equal to x and is denoted as $\lfloor x \rfloor$.

   For example, $\lfloor 3.2 \rfloor = \lfloor 3.8 \rfloor = 3$ and $\lfloor -3.2 \rfloor = \lfloor -3.9 \rfloor = -4$.

6. *median* of a set of real numbers $\{x_1, x_2, ..., x_N\}$ is obtained by rank ordering the numbers in the set and choosing the middle number in the ordered set.

   For example, the median of $\{7, 13, 1, 6, 3\}$ is 6 and the median of $\{7, 13, 1, 6, 3, 9\}$ is $(6 + 7)/2 = 6.5$.

7. *Dirac delta* function $\delta(\tau)$ is a function of τ with infinite height, zero width, and unit area. It is the limiting form of a unit area pulse function
$$p_\triangle(\tau) = \begin{cases} \frac{1}{2\triangle}, & -\triangle < \tau \leq \triangle \\ 0, & \text{elsewhere} \end{cases}$$
   as $\triangle$ goes to 0, i.e.,
$$\lim_{\triangle \to 0} \int_{-\infty}^{\infty} p_\triangle(\tau)\, d\omega = \int_{-\infty}^{\infty} \delta(\tau)\, d\tau = 1. \qquad \text{(R2.1)}$$

Equation(R2.1) also holds even when we reverse the direction of axis τ and shift $\delta(-\tau)$ by an amount t, i.e.,

$$\int_{-\infty}^{\infty} \delta(t-\tau)\, d\tau = 1. \tag{R2.2}$$

Because of the above properties, we have

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau = x(\tau)|_{\tau=t} = x(t). \tag{R2.3}$$

Equation(R2.3) holds for any value of t, and it is referred as the *sifting property* of the Dirac delta function.

8. The *modulo* operation of integer X over integer N is the residue of X divided by N:

$$\langle X\rangle_N = X - kN, k = \lfloor X/N \rfloor.$$

It can be verified that the modulo operation is *linear*. When negative numbers are used, $\langle X\rangle_N$ has the same sign as N. For example, $\langle 67\rangle_{13} = 67 - 5 \cdot 13 = 2$, $\langle 67\rangle_{-13} = 67 - (-13)\cdot(-6) = -11$ and $\langle -67\rangle_{13} = -67 - 13\cdot(-6) = 11$.

The statement "X is congruent to Y, modulo N" means that

$$\langle X\rangle_N = \langle Y\rangle_N.$$

The notation

$$\langle X^{-1}\rangle_N$$

denotes the multiplicative inverse of X evaluated modulo N, i.e., if $\langle X^{-1}\rangle_N = \alpha$, then $\langle X\alpha\rangle_N = 1$. For example, $\langle 3^{-1}\rangle_4 = 3$ because $\langle 3\cdot 3\rangle_4 = 1$, and $\langle 8^{-1}\rangle_5 = 2$ because $\langle 8\cdot 2\rangle_5 = 1$.

In the case of polynomial, the operation $a(z)$ mod $b(z)$ is the residue $r(z)$ after the polynomial division $a(z)/b(z)$. For example, if $a(z) = 4z^{-3}+2z^{-2}+5z^{-1}+1$ and $b(z) = z^{-2}+3z^{-1}+4$ then the residue after the division

$$\frac{a(z)}{b(z)} = 4z^{-1} - 10 + \frac{19z^{-1}+41}{z^{-2}+3z^{-1}+4}$$

is $19z^{-1}+41$. Therefore, $a(z) \bmod b(z) = r(z) = 19z^{-1}+41$.

R3 Commonly Used Differentials and Integrals

R3.1 Differentials

$$d(uv) = u\,dv + v\,du \tag{R3.1}$$
$$d(\frac{u}{v}) = \frac{v\,du - u\,dv}{v^2} \tag{R3.2}$$
$$d(u^n) = n\,u^{n-1}\,du \tag{R3.3}$$
$$d\,e^u = e^u\,du \tag{R3.4}$$
$$d\,a^u = (a^u \log_e a)\,du \tag{R3.5}$$
$$d(\log_e u) = u^{-1}\,du \tag{R3.6}$$
$$d\,\sin u = \cos u\,du \tag{R3.7}$$
$$d\,\cos u = -\sin u\,du. \tag{R3.8}$$

R3.2 Integrals

$$\int f(g(x))g'(x)\,dx = \int f(y)\,dy,\; y = g(x) \text{ and } g'(x) = dy/dx \tag{R3.9}$$
$$\int u\,dv = uv - \int v\,du \tag{R3.10}$$
$$\int \frac{f'(x)\,dx}{f(x)} = \log_e f(x) \tag{R3.11}$$
$$\int \frac{dx}{x} = \log_e x \tag{R3.12}$$

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} \tag{R3.13}$$

$$\int e^x\, dx = e^x \tag{R3.14}$$

$$\int a^x\, dx = \frac{a^x}{\log_e a} \tag{R3.15}$$

$$\int a^{bx}\, dx = \frac{a^{bx}}{b \log_e a} \tag{R3.16}$$

$$\int \log_e x\, dx = x\, \log_e x - x \tag{R3.17}$$

$$\int \sin x\, dx = -\cos x \tag{R3.18}$$

$$\int \cos x\, dx = \sin x \tag{R3.19}$$

$$\int \tan x\, dx = -\log \cos x. \tag{R3.20}$$

R3.3 l'Hôpital's Rule

Consider a fraction $f(x)/g(x)$ for which at $x = x_0$, $f(x_0) = g(x_0) = 0$ (or $f(x_0) = g(x_0) = \infty$). Then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \tag{R3.21}$$

as long as the limits on the right-hand side exist and are finite.

R3.4 Examples

Example R3.1. Evaluate the integral

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n}\, d\omega,$$

where

$$X(\omega) = \begin{cases} \cos(\alpha\omega), & |\omega| \le \omega_0 \\ 0, & \omega_0 < |\omega| \le \pi. \end{cases}$$

Answer:

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} \cos(\alpha\omega)e^{j\omega n}\,d\omega \\
&= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} \frac{1}{2}(e^{j\alpha\omega}+e^{-j\alpha\omega})e^{j\omega n}\,d\omega \\
&= \frac{1}{4\pi}\Big(\int_{-\omega_0}^{\omega_0} e^{j\alpha\omega}e^{j\omega n}\,d\omega + \int_{-\omega_0}^{\omega_0} e^{-j\alpha\omega}e^{j\omega n}\,d\omega\Big) \\
&= \frac{1}{4\pi}\Big(\frac{1}{j(\alpha+n)}e^{j(\alpha+n)\omega}\Big|_{-\omega_0}^{\omega_0} + \frac{1}{j(-\alpha+n)}e^{j(-\alpha+n)\omega}\Big|_{-\omega_0}^{\omega_0}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4\pi}\Big(\frac{1}{j(\alpha+n)}2j\sin(\alpha+n)\omega_0 + \frac{1}{j(-\alpha+n)}2j\sin(-\alpha+n)\omega_0\Big) \\
&= \frac{\sin(\alpha+n)\omega_0}{2\pi(\alpha+n)} + \frac{\sin(-\alpha+n)\omega_0}{2\pi(-\alpha+n)}.
\end{aligned}
$$

Example R3.2. Evaluate the integral

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}\,d\omega,$$

where

$$X(\omega) = \begin{cases} \omega, & |\omega| \le \omega_0 \\ 0, & \omega_0 < |\omega| \le \pi. \end{cases}$$

using integration by parts.

Answer:

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} \omega e^{j\omega n}\,d\omega \\
&= \frac{1}{2\pi}\cdot\frac{1}{jn}\int_{-\omega_0}^{\omega_0} \omega e^{j\omega n}\,d(jn)\omega \\
&= \frac{1}{2\pi}\cdot\frac{1}{jn}\int_{-\omega_0}^{\omega_0} \omega\,de^{j\omega n} \\
&= \frac{1}{2\pi}\cdot\frac{1}{jn}\Big[\omega e^{j\omega n}\Big|_{-\omega_0}^{\omega_0} - \int_{-\omega_0}^{\omega_0} e^{j\omega n}\,d\omega\Big] \\
&= \frac{1}{2\pi}\cdot\frac{1}{jn}\Big[\omega_0 e^{j\omega_0 n} - (-\omega_0)e^{-j\omega_0 n} - \frac{1}{jn}e^{j\omega n}\Big|_{-\omega_0}^{\omega_0}\Big] \\
&= \frac{1}{2\pi}\cdot\frac{1}{jn}\Big[\omega_0(e^{j\omega_0 n} + e^{-j\omega_0 n}) - \frac{1}{jn}(e^{j\omega_0 n} - e^{-j\omega_0 n})\Big] \\
&= \frac{1}{\pi(jn)}\Big[\omega_0\cos(\omega_0 n) - \frac{1}{n}\sin(\omega_0 n)\Big].
\end{aligned}
$$

R4 Complex Numbers

R4.1 Definition

A complex number z is represented in the Cartesian coordinate as

$$z = x + jy$$

where $j = \sqrt{-1}$, and x and y are real numbers and denoted as the real and imaginary parts of z, respectively. The complex numbers can also be represented in polar form as

$$z = |z|e^{j\theta},$$

where $|z|$ and θ are the magnitude and angle of z, respectively:

$$|z| = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}(\frac{y}{x}).$$

The *principal value* of the angle of z is given by

$$-\pi < \theta \leq \pi.$$

Figure R4.1 shows the representation of a complex number z in the complex plane. By using *Euler's Formula* (see Eq.(R1.29)) we can find the representation of a complex number in Cartesian form from its polar form:

$$x = |z|\cos\theta, \quad y = |z|\sin\theta.$$

It should be pointed out here that negative real numbers have angle

$$\theta = (2k + 1)\pi$$

with k any integer.

Example R4.1. Let $z = 2 + j\sqrt{3}$, we can express it in polar form by calculating its magnitude and angle:

$$|z| = \sqrt{2^2 + 3} = \sqrt{7}, \quad \theta = \tan^{-1}(\frac{\sqrt{3}}{2}).$$

Therefore,

$$z = 2 + j\sqrt{3} = \sqrt{7}e^{j\tan^{-1}(\frac{\sqrt{3}}{2})}.$$

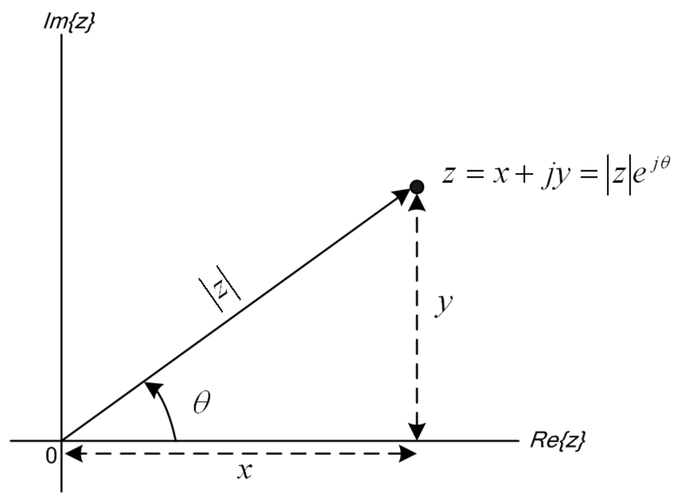

Figure R4.1: Representation of a complex number z in Cartesian form and polar form.

The *conjugate* of a complex number in Cartesian form is obtained by negating the imaginary part:

$$z^* = (x + jy)^* = x^* + (jy)^* = x - jy.$$

In polar form, the conjugate is obtained by changing the sign of the angle:

$$z^* = (re^{j\theta})^* = r\,e^{-j\theta}.$$

R4.2 Complex Arithmetic

(1) Addition and Subtraction

$z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$ be two complex numbers. Then

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

where $(x_1 + x_2)$ are $(y_1 + y_2)$ are the real and imaginary parts of the sum $z_1 + z_2$, respectively. Similarly,

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

where $(x_1 - x_2)$ are $(y_1 - y_2)$ are the real and imaginary parts of the difference $z_1 - z_2$, respectively.

Example R4.2. Let $z_1 = 1.3 + j5.2$ and $z_2 = 2.7 - j3.6$ then

$$\begin{aligned} z_1 + z_2 &= (1.3 + j5.2) + (2.7 - j3.6) = 4 + j1.6 \\ z_1 - z_2 &= (1.3 + j5.2) - (2.7 - j3.6) = -1.4 + 8.8j. \end{aligned}$$

(2) Multiplication

Let $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$ then

$$\begin{aligned} z_1 \cdot z_2 &= (x_1 + jy_1)(x_2 + jy_2) \\ &= x_1x_2 + jx_1y_2 + jx_2y_1 + j^2y_1y_2 \\ &= (x_1x_2 - y_1y_2) + j(x_1y_2 + x_2y_1). \end{aligned}$$

Example R4.3. Let $z_1 = 1 + j\sqrt{3}, z_2 = 2 - j2$. The product of z_1, z_2 calculated in polar form is given by

$$(1 + j\sqrt{3})(2 - j2) = 2e^{j\pi/3} \cdot 2\sqrt{2}e^{-j\pi/4} = 4\sqrt{2}e^{j\pi/12} = 5.4641 + j1.4641.$$

Calculating in the Cartesian form we get

$$(1 + j\sqrt{3})(2 - j2) = (2 + 2\sqrt{3}) + j(2\sqrt{3} - 2) = 5.4641 + j1.4641.$$

(3) Division

The division of two complex numbers z_0 and z_1 can be carried out either in polar form or in Cartesian form. In the former case

$$w = \frac{z_0}{z_1} = \frac{r_0e^{j\theta_0}}{r_1e^{j\theta_1}} = \frac{r_0}{r_1}e^{j(\theta_0-\theta_1)}.$$

In the latter case

$$w = \frac{z_0}{z_1} = \frac{x_0 + jy_0}{x_1 + jy_1} = \frac{(x_0 + jy_0)(x_1 - jy_1)}{(x_1 + jy_1)(x_1 - jy_1)} = \frac{(x_0x_1 + y_0y_1) + j(x_1y_0 - x_0y_1)}{x_1^2 + y_1^2}.$$

Example R4.4. To divide $2 + j2$ by $1 - j$, we calculate in polar form as follows:

$$\frac{2 + j2}{1 - j} = \frac{2\sqrt{2}e^{j\frac{\pi}{4}}}{\sqrt{2}e^{-j\frac{\pi}{4}}} = 2e^{j(\frac{\pi}{4})-(-\frac{\pi}{4})} = 2e^{j\frac{\pi}{2}} = j2.$$

Calculating in the Cartesian form we get

$$\frac{2 + j2}{1 - j} = \frac{(2 + j2)(1 + j)}{(1 - j)(1 + j)} = \frac{(2 - 2) + j(2 + 2)}{1^2 + 1^2} = j2.$$

(4) Inverse

The inverse of a complex number is a special case of division where the numerator is 1. In polar form we have

$$z^{-1} = \frac{1}{z} = \frac{1}{re^{j\theta}} = \frac{1}{r}e^{-j\theta}.$$

Equivalently, in the Cartesian form we have

$$z^{-1} = \frac{1}{z} = \frac{1}{x+jy} = \frac{x-jy}{(x+jy)(x-jy)} = \frac{x-jy}{x^2+y^2}.$$

R5 Complex Variables

R5.1 Function of a Complex Variable

A function of the complex variable z can be written as

$$f(z) = u(z) + jv(z)$$

where $u(z)$ and $v(z)$ are real functions of z. In the Cartesian form, we define $z = x + jy$ for real x and y. Therefore, the values of $u(z)$ and $v(z)$ depend on x and y, and we can express the complex function $f(z)$ as

$$f(z) = u(x, y) + jv(x, y).$$

If $z = re^{j\theta}$, then $f(z)$ can be expressed as

$$f(z) = u(r, \theta) + jv(r, \theta),$$

where $u(r, \theta)$ and $v(r, \theta)$ are the real and imaginary parts of $f(z)$.

R5.2 Analytic Function of a Complex Variable

Definition R5.1. A function $f(z)$ is said to be *differentiable* at a point z_0 in the z-plane if the limit

$$f'(z_0) = \lim_{\triangle z \to 0} \frac{f(z_0 + \triangle z) - f(z_0)}{\triangle z}$$

exists. Note that $f(z_0 + \triangle z)$ can approach $f(z_0)$ along any path. This limit is called the *derivative* of $f(z)$ at point z_0.

Definition R5.2. A function $f(z)$ of a complex variable z is *analytic* in the region R in the complex z-plane if and only if all the derivatives of $f(z)$ exist at all points inside the region R.

R5.3 Analytic Continuation

If the values of a function $f(z)$ of a complex variable are known everywhere on a closed contour C inside a region R where $f(z)$ is analytic, then the values of $f(z)$ at all points in R can be found by mapping from the contour C to any point in R.

R5.4 Cauchy's Integral Formula

If a function $f(z)$ is analytic both on and inside a counterclockwise closed contour C and if z_0 is any point inside C, then

$$f(z_0) = \frac{1}{2\pi j} \oint_C f(z) \frac{1}{z - z_0}\, dz, \tag{R5.1}$$

$$f'(z_0) = \frac{1}{2\pi j} \oint_C f(z) \frac{1}{(z - z_0)^2}\, dz \tag{R5.2}$$

$$f''(z_0) = \frac{2}{2\pi j} \oint_C f(z) \frac{1}{(z - z_0)^3}\, dz \tag{R5.3}$$

$$\vdots \tag{R5.4}$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi j} \oint_C f(z) \frac{1}{(z - z_0)^{n+1}}\, dz. \tag{R5.5}$$

where $f'(z_0) = \left.\frac{d\,f(z)}{dz}\right|_{z=z_0}$, $f''(z_0) = \left.\frac{d^2\,f(z)}{dz^2}\right|_{z=z_0}$, and $f^{(n)}(z_0) = \left.\frac{d^n\,f(z)}{dz^n}\right|_{z=z_0}$.

Eq.(R5.1) is often referred as the Cauchy's integral formula.

By combining Eq.(R5.1) - Eq.(R5.5) we arrive at an useful relation:

$$\frac{1}{2\pi j} \oint_C z^{k-1}\, dz = \begin{cases} 1 & , k = 0 \\ 0 & , k \neq 0 \end{cases} \tag{R5.6}$$

where C is a counterclockwise closed contour encircling $z = 0$.

R5.5 Cauchy's Residue Theorem

If a function $f(z)$ is analytic both on and inside a counterclockwise closed contour C except at poles z_k, $k = 1, 2, ..., n$, then

$$\frac{1}{2\pi j} \oint_C f(z)\, dz = \sum_k \Big[\text{residue of } f(z) \text{ at pole } z_k \text{ inside } C\Big]. \tag{R5.7}$$

In the case when $f(z)$ is a rational function of z and has pole at $z = z_k$ of multiplicity m, we can express $f(z)$ as

$$f(z) = \frac{\Gamma(z)}{(z - z_k)^m},$$

where $\Gamma(z)$ does not have any pole at $z = z_k$. Thus the residue of $f(z)$ at the pole z_k inside C is given by

$$\frac{1}{(m-1)!}\left[\frac{d^{m-1}\Gamma(z)}{d\,z^{m-1}}\right]_{z=z_k}. \tag{R5.8}$$

R6 Continuous-Time Signals

R6.1 Energy and Power

The *total energy* of a continuous-time signal $x(t)$ is given by

$$E_x = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2\, dt.$$

The *average power* of a continuous-time $x(t)$ is given by

$$P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt.$$

The definition of total energy can explained as the area under the squared signal $|x(t)|^2$, and it is a measurement of the strength of the signal $x(t)$ over infinite time. However, there are signals with infinite energy so we need to evaluate the average power of the signal $x(t)$ as a measurement of the strength over one unit time.

R6.2 Continuous-Time Sinusoidal and Exponential Signals

R6.2.1 Definition

The continuous-time *real sinusoidal signal* with constant amplitude is of the form

$$x(t) = A\cos(\Omega_0 t + \phi), \tag{R6.1}$$

where A, Ω_0 and ϕ are real numbers. The parameters A, Ω_0 and ϕ are called, respectively, the *amplitude*, the *angular frequency*, and the *phase* of the sinusoidal signal $x(t)$.

The *complex exponential signal* is expressed in the form

$$x(t) = A\alpha^t, \tag{R6.2}$$

where

$$\alpha = e^{\sigma_0 + j\Omega_0}, A = |A|\, e^{j\phi}. \tag{R6.3}$$

If A and α are both real, the signal of Eq. (R6.2) reduces to *real exponential signal*. For $t \geq 0$ such a signal with $|\alpha| < 1$ decays exponentially as t increases and with $|\alpha| > 1$ grows exponentially as t increases.

In addition, we can rewrite Eq. (R6.2) as

$$\begin{aligned} x(t) &= Ae^{(\sigma_0 + j\Omega_0)t} = |A|\, e^{\sigma_0 t}\, e^{j(\Omega_0 t + \phi)} && \text{(R6.4)} \\ &= |A|\, e^{\sigma_0 t}\, \cos(\Omega_0 t + \phi) + j|A|\, e^{\sigma_0 t}\, \sin(\Omega_0 t + \phi). && \text{(R6.5)} \end{aligned}$$

Thus the real and imaginary parts of a complex exponential signal are real sinusoidal signals.

The *fundamental period* T_0 of a complex exponential signal (Eq. (R6.4)) with $\sigma_0 = 0$ is defined to be the smallest positive T_0 satisfying

$$|A|\, e^{j(\Omega_0 t + \phi)} = |A|\, e^{j(\Omega_0 (t + T_0) + \phi)}, \tag{R6.6}$$

or equivalently,

$$e^{j\Omega_0 T_0} = 1. \tag{R6.7}$$

Therefore,

$$T_0 = \frac{2\pi}{|\Omega_0|}. \tag{R6.8}$$

R6.2.2 Properties

The properties of continuous-time sinusoidal and exponential signals and comparisons with discrete-time sinusoidal and exponential sequences are discussed as follows.

1. Periodicity for any choice of Ω_0
   Note that the continuous-time sinusoidal signal $A\cos(\Omega_0 t + \phi)$ (Eq. (R6.1)) and the continuous-time complex exponential signal $|A|\, e^{j(\Omega_0 t+\phi)}$ are periodic signals of any choice of Ω_0. However, discrete-time sequences are not always periodic with any choice of ω_0. Discrete sinusoidal sequence $A\cos(\omega_0 n + \phi)$ and discrete complex exponential sequence $|A|\, e^{j(\omega_0 n+\phi)}$ are periodic with period N only if $\omega_0 N$ is an integer multiple of 2π, i.e., $\omega_0 N = 2\pi r$ where N and r are positive integers. For example, $\cos(\frac{\pi n}{4})$ is a periodic sequence while $\cos(\frac{n}{4})$ is not periodic.

2. Distinctness for different Ω_0, Ω_1
   Any two continuous-time sinusoidal signals

   $$A\cos(\Omega_0 t + \phi),\ A\cos(\Omega_1 t + \phi),\ \Omega_0 \neq \Omega_1$$

   have different waveforms. Similarly, any two continuous-time exponential signals with $\Omega_0 \neq \Omega_1$ also have different waveforms. Unlike the continuous-time case, discrete-time sinusoidal sequences

   $$A\cos(\omega_0 n + \phi),\ A\cos(\omega_1 n + \phi),\ \omega_0 = \omega_1 + 2\pi k$$

   have the same sequence values. Similarly, any two discrete-time exponential sequences with $\omega_0 = \omega_1 + 2\pi k$ also have the same sequence values.

R6.3 Continuous-Time Eigenfunction

If the input signal of any LTI system has output signal being the input signal multiplied by a complex constant, this certain type of input signal is called the *eigenfunction* and the complex constant is called the *eigenvalue*.

Example R6.1. We want to show that the complex exponential signal defined in Eq. (R6.2)

$$x(t) = A\alpha^t$$

is an eigenfunction of an LTI continuous-time system with an impulse response $h(t)$.

By using the convolution integral, we have

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau) A\alpha^{(t-\tau)}\, d\tau \\ &= \Big(\int_{-\infty}^{\infty} h(\tau)\alpha^{-\tau}\, d\tau\Big) A\alpha^t. \end{aligned}$$

Since the integral inside the brackets is independent of t, we can therefore say that the input signal $A\alpha^t$ is an eigenfunction.

Example R6.2. We want to show that the sum of any two complex exponential signals

$$x(t) = A\alpha^t + B\beta^t$$

is not an eigenfunction of an LTI continuous-time system with an impulse response $h(t)$.

By using the convolution integral, we have

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)\Big(A\alpha^{(t-\tau)} + B\beta^{(t-\tau)}\Big)\, d\tau \\ &= \Big(\int_{-\infty}^{\infty} h(\tau)\alpha^{-\tau}\, d\tau\Big) A\alpha^t + \Big(\int_{-\infty}^{\infty} h(\tau)\beta^{-\tau}\, d\tau\Big) B\beta^t. \end{aligned}$$

Since the input signal $x(t)$ cannot be extracted from the summation above, we can therefore say that the input signal $A\alpha^t + B\beta^t$ is not an eigenfunction.

R6.4 Continuous-Time Fourier Series

R6.4.1 Definition

Given a periodic continuous-time signal $x(t)$ with period T_0 and fundamental frequency $\Omega_0 = 2\pi/T_0$, the Fourier series expansion of $x(t)$ is given by the linear combination of the set of harmonically related complex exponentials

$$e^{jk\Omega_0 t} = e^{jk\frac{2\pi}{T_0}t}, \quad k = 0, \pm1, \pm2, ...,$$

i.e.,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T_0}t} \tag{R6.9}$$

$$a_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\Omega_0 t}\, dt = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\frac{2\pi}{T_0}t}\, dt. \tag{R6.10}$$

Note that the notation $\int_{T_0}$ denotes the integration over any interval of length T_0. The Eq.(R6.9) is referred to as the *synthesis* equation and the Eq.(R6.10) is referred to as the *analysis* equation. The coefficient a_k is called the *Fourier series coefficient*.

Table R6.1: Properties of Continuous-Time Fourier Series.

Type of Property	**Periodic Signal with frequency $\Omega_0 = 2\pi/T$**	**Fourier Series Coefficients**
	$g(t)$ $h(t)$	a_k b_k
Linearity	$\alpha g(t) + \beta h(t)$	$\alpha a_k + \beta b_k$
Time Shifting	$g(t - t_0)$	$a_k e^{-jk\Omega_0 t_0}$
Frequency Shifting	$e^{jM\Omega_0 t} g(t)$	a_{k-M}
Multiplication	$g(t)h(t)$	$\sum_{l=-\infty}^{\infty} a_l b_{k-l}$
Time Reversal	$g(-t)$	a_{-k}
Conjugation	$g^*(t)$	a^*_{-k}
Time Scaling	$g(\alpha t), \alpha > 0$	a_k
Periodic Convolution	$\int_T g(\tau)h(t-\tau)\,d\tau$	$Ta_k b_k$

Example R6.3. Find the Fourier series coefficients of the continuous-time signal

$$x(t) = 1 + \cos(\Omega_0 t) + 2\cos(2\Omega_0 t + \frac{\pi}{3}) + 4\sin(3\Omega_0 t - \frac{\pi}{4})$$

with fundamental frequency Ω_0.

By using the Euler's Formula, it can be shown that

$$\begin{aligned} x(t) &= 1 + \frac{1}{2}(e^{j\Omega_0 t} + e^{-j\Omega_0 t}) + \frac{2}{2}(e^{j(2\Omega_0 t + \frac{\pi}{3})} + e^{-j(2\Omega_0 t + \frac{\pi}{3})}) + \frac{4}{2j}(e^{j(3\Omega_0 t - \frac{\pi}{4})} - e^{-j(3\Omega_0 t - \frac{\pi}{4})}) \\ &= 1 + \frac{1}{2}e^{j\Omega_0 t} + \frac{1}{2}e^{-j\Omega_0 t} + e^{j\frac{\pi}{3}}e^{j2\Omega_0 t} + e^{-j\frac{\pi}{3}}e^{-j2\Omega_0 t} + \frac{2}{j}e^{-j\frac{\pi}{4}}e^{j3\Omega_0 t} - \frac{2}{j}e^{j\frac{\pi}{4}}e^{-j3\Omega_0 t}. \end{aligned}$$

Therefore, the Fourier series coefficients are

$$
\begin{aligned}
&a_0 = 1, \\
&a_1 = \frac{1}{2},\ a_{-1} = \frac{1}{2}, \\
&a_2 = e^{j\frac{\pi}{3}} = \frac{1+j\sqrt{3}}{2},\ a_{-2} = e^{-j\frac{\pi}{3}} = \frac{1-j\sqrt{3}}{2}, \\
&a_3 = \frac{2}{j}e^{-j\frac{\pi}{4}} = \sqrt{2}(-1-j),\ a_{-3} = -\frac{2}{j}e^{j\frac{\pi}{4}} = \sqrt{2}(-1+j), \\
&a_k = 0, |k| > 3.
\end{aligned}
$$

Example R6.4. Find the Fourier series coefficients of the impulse train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$$

with period T_0.

By calculating Eq.(R6.10) in the interval $-T_0/2 \leq t \leq T_0/2$, we can get

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)e^{-jk\frac{2\pi}{T_0}t}\,dt = \frac{1}{T_0}.$$

Therefore, all the Fourier series coefficients of the impulse train have the same value $1/T_0$.

Some important properties of continuous-time Fourier series are listed in Table R6.1 for quick reference.

R6.4.2 Dirichlet Conditions

In order to verify the existence of Fourier series representation for a periodic signal $x(t)$, we need to examine the Dirichlet conditions. The Dirichlet conditions are given by:

1. $x(t)$ must be absolutely integrable, i.e.,

$$\int_{T_0} |x(t)|\,dt < \infty$$

2. In any finite interval of time, $x(t)$ has a finite number of local maxima and local minima.

3. In any finite interval of time, $x(t)$ has a finite number of discontinuities.

The Dirichlet conditions guarantee that $x(t)$ equals its Fourier series representation

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$$

at all values of t except at discontinuities of $x(t)$. Note that Dirichlet conditions are only sufficient but not necessary conditions.

R6.5 Continuous-Time Fourier Transform

Given an aperiodic continuous-time signal $x(t)$, the continuous-time Fourier transform of $x(t)$ is given by

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t}\, dt \tag{R6.11}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t}\, d\Omega. \tag{R6.12}$$

The transform $X(j\Omega)$ is referred to as the *spectrum* of $x(t)$ because it provides the information of $x(t)$ when evaluated by complex exponential signals at different frequencies.

Some important properties of continuous-time Fourier transform are listed in Table R6.2 for quick reference.

Table R6.2: Properties of Continuous-Time Fourier Transform.

Property	Signal	Fourier Transform
	$g(t)$	$G(j\Omega)$
	$h(t)$	$H(j\Omega)$
Linearity	$\alpha g(t) + \beta h(t)$	$\alpha G(j\Omega) + \beta H(j\Omega)$
Time Shifting	$g(t - t_0)$	$G(j\Omega)e^{-j\Omega t_0}$
Frequency Shifting	$e^{j\Omega_0 t}g(t)$	$G(j(\Omega - \Omega_0))$
Multiplication	$g(t)h(t)$	$\frac{1}{2\pi}G(j\Omega) * H(j\Omega)$
Time Reversal	$g(-t)$	$G(-j\Omega)$
Conjugation	$g^*(t)$	$G^*(-j\Omega)$
Time Scaling	$g(\alpha t)$	$\frac{1}{\lvert\alpha\rvert}G\left(\frac{j\Omega}{\alpha}\right)$
Convolution	$g(t) * h(t)$	$G(j\Omega)H(j\Omega)$
Differentiation in Time	$\frac{d}{dt}g(t)$	$j\Omega G(j\Omega)$
Integration	$\int_{-\infty}^{t} g(\tau)\, d\tau$	$\frac{1}{j\Omega}G(j\Omega) + \pi G(0)\delta(\Omega)$
Real and Even in Time	$g(t)$ real and even	$G(j\Omega)$ real and even
Real and Odd in Time	$g(t)$ real and odd	$G(j\Omega)$ purely imaginary and odd

R7 Discrete Fourier Series

Given a periodic sequence $x[n]$ with period N, the fundamental period is defined to be the smallest integer N such that $x[n] = x[n + N]$ is satisfied, and the fundamental frequency is defined to be $\omega_0 = 2\pi/N$. The harmonics are sequences whose frequencies are integer multiples of the fundamental frequency. For discrete complex exponential signals, the $k - th$ *harmonic* is expressed as

$$e^{jk\omega_0 n} = e^{jk\frac{2\pi}{N}n},\ k = 0, \pm 1, \pm 2, \cdots .$$

Note that there are only N distinct harmonics for discrete complex exponential signals with fundamental frequency $\omega_0 = 2\pi/N$ because every two signals with frequencies which differ in $2\pi m$ have the same waveform, i.e,

$$e^{jk(\omega_0+2m\pi)n} = e^{jk(\frac{2\pi}{N}+2m\pi)n} = e^{jk(\frac{2\pi}{N})n} \cdot e^{jk2mn\pi} = e^{jk(\frac{2\pi}{N})n}.$$

The discrete Fourier series expansion of periodic signal $x[n]$ is the expression in form of a linearly weighted combination of a fundamental and a series of harmonic complex exponential signals.

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} = \sum_{k=0}^{N-1} a_k e^{jk\frac{2\pi}{N}n},$$

where

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n}.$$

Example R7.1. Calculate the Fourier series coefficients a_k for the following periodic signal

$$\{x[n]\} = \{..., \underset{\uparrow}{1}, 1, 1, 1, 0, 0, ...\}.$$

We can observe that $N = 6$ so

$$\begin{aligned}
a_k &= \tfrac{1}{6}\textstyle\sum_{n=0}^{5} x[n]e^{-jk(\frac{2\pi}{6})6} \\
&= \tfrac{1}{6}(e^{-jk(\frac{2\pi}{6})0} + e^{-jk(\frac{2\pi}{6})1} + e^{-jk(\frac{2\pi}{6})2} + e^{-jk(\frac{2\pi}{6})3} + 0 + 0) \\
&= \tfrac{1}{6}(1 + e^{-jk\frac{\pi}{3}} + e^{-jk\frac{2\pi}{3}} + e^{-jk\pi}) \\
&= \tfrac{1}{6}(1 + e^{-jk\frac{\pi}{3}} + (-1)^k e^{jk\frac{\pi}{3}} + (-1)^k).
\end{aligned}$$

Example R7.2. Calculate signal $x[n]$ from the following Fourier series coefficients a_k

$$\{a_k\} = \{..., 1/4, 1/2, \underset{\uparrow.}{1}, 1/2, 1/4, 0, 1/4, 1/2, 1, 1/2...\}.$$

We can observe that $N = 6$ so

$$\begin{aligned}
x[n] &= \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{6}n} = 1 + \frac{1}{2}e^{j\frac{2\pi}{6}n} + \frac{1}{4}e^{j\frac{2\pi}{6}2n} + 0 + \frac{1}{4}e^{j\frac{2\pi}{6}4n} + \frac{1}{2}e^{j\frac{2\pi}{6}5n} \\
&= 1 + \frac{1}{2}e^{j\frac{\pi n}{3}} + \frac{1}{4}e^{j\frac{2\pi n}{3}} + 0 + \frac{1}{4}e^{j\frac{4\pi n}{3}} + \frac{1}{2}e^{j\frac{5\pi n}{3}} \\
&= 1 + \frac{1}{2}e^{j\frac{\pi n}{3}} + \frac{1}{4}e^{j\frac{2\pi n}{3}} + 0 + \frac{1}{4}e^{j(2\pi n - \frac{2\pi n}{3})} + \frac{1}{2}e^{j(2\pi n - \frac{\pi n}{3})} \\
&= 1 + \frac{1}{2}e^{j\frac{\pi n}{3}} + \frac{1}{4}e^{j\frac{2\pi n}{3}} + 0 + \frac{1}{4}e^{-j\frac{2\pi n}{3}} + \frac{1}{2}e^{-j\frac{\pi n}{3}} \\
&= 1 + \cos(\frac{\pi n}{3}) + \frac{1}{2}\cos(\frac{2\pi n}{3}).
\end{aligned}$$

R8 Matrix Algebra

R8.1 Definition

A matrix is a rectangular array of real or complex numbers enclosed in brackets; for instance,

$$\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 5 & 6 \\ 2 & 1 & -3 \end{bmatrix}, \begin{bmatrix} 3j & 5 \\ 2-4j & 1+j \\ 7 & 5-3j \end{bmatrix}.$$

A matrix with K rows and M columns is called a $K \times M$ matrix. For example, the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a 3×3 matrix. The matrix

$$\begin{bmatrix} 1 & 4 \\ 2 & 6 \\ -3j & 1 \\ 1+j & 1 \end{bmatrix}$$

is a 4×2 matrix. The matrix

$$\mathbf{U} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1M} \\ a_{21} & a_{22} & \cdots & a_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ a_{K1} & a_{K2} & \cdots & a_{KM} \end{bmatrix} \tag{R8.1}$$

is a $K \times M$ matrix and the number a_{rs}, $r = 1,\ 2, ...,\ K$, and $s = 1,\ 2, ...,\ M$ is called the entry of $\mathbf{U}$.

R8.2 Transpose

The transpose, $\mathbf{U}^T$, of a $K \times M$ matrix $\mathbf{U}$ is the $M \times K$ matrix formed by interchanging the rows and columns of $\mathbf{U}$. For example, the transpose of the matrix $\mathbf{U}$ given in Eq.(R8.1) is a $M \times K$ matrix given by

$$\mathbf{U}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{K1} \\ a_{12} & a_{22} & \cdots & a_{K2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1M} & a_{2M} & \cdots & a_{KM} \end{bmatrix}. \tag{R8.2}$$

R8.3 Toeplitz Matrix

The $N \times N$ matrix $\mathbf{U}$ is a Toeplitz matrix if all entries along the line parallel to the main diagonal are the same. For example,

$$\mathbf{U} = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} \\ a_1 & a_0 & a_{-1} & a_{-2} \\ a_2 & a_1 & a_0 & a_{-1} \\ a_3 & a_2 & a_1 & a_0 \end{bmatrix}$$

is a 4×4 Toeplitz matrix.

R8.4 Circulant Matrix

The $N \times N$ matrix $\mathbf{U}$ is a circulant matrix if each row equals the right circular shift of the previous row by one entry. For example,

$$\mathbf{U} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_3 & a_0 & a_1 & a_2 \\ a_2 & a_3 & a_0 & a_1 \\ a_1 & a_2 & a_3 & a_0 \end{bmatrix}$$

is a 4×4 Circulant matrix.

R8.5 Determinant

If the 2×2 matrix $\mathbf{U}$ is

$$\mathbf{U} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \tag{R8.3}$$

then the determinant of the $\mathbf{U}$ is given by

$$\det(\mathbf{U}) = a_{11}a_{22} - a_{12}a_{21}. \tag{R8.4}$$

Example R8.1. The determinant of the matrix

$$\mathbf{U} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

is $\det(\mathbf{U}) = 3 \cdot 2 - 4 \cdot 1 = 6 - 4 = 2$.

If the 3×3 matrix $\mathbf{U}$ is

$$\mathbf{U} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \tag{R8.5}$$

then the determinant of $\mathbf{U}$ is given by

$$\det(\mathbf{U}) = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33}. \tag{R8.6}$$

Example R8.2. The determinant of the matrix

$$\begin{bmatrix} 1 & 4 & -6 \\ 2 & 1 & 3 \\ 4 & 5 & -2 \end{bmatrix}$$

is

$$\begin{aligned} \det(\mathbf{U}) &= 1 \cdot 1 \cdot (-2) + 2 \cdot 5 \cdot (-6) + 4 \cdot 3 \cdot 4 - (-6) \cdot 1 \cdot 4 - 1 \cdot 5 \cdot 3 - 4 \cdot 2 \cdot (-2) \\ &= (-2) + (-60) + 48 - (-24) - 15 - (-16) = 11. \end{aligned}$$

If the $N \times N$ matrix $\mathbf{U}$ is

$$\mathbf{U} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}, \tag{R8.7}$$

then the determinant of $\mathbf{U}$ is

$$\begin{aligned} \det(\mathbf{U}) &= a_{r1}(-1)^{r+1}M_{r1} + a_{r2}(-1)^{r+2}M_{r2} + \cdots + a_{rN}(-1)^{r+N}M_{rN} & \text{(R8.8)} \\ &= a_{1s}(-1)^{1+s}M_{1s} + a_{2s}(-1)^{2+s}M_{2s} + \cdots + a_{Ns}(-1)^{N+s}M_{Ns} & \text{(R8.9)} \end{aligned}$$

where $r, s = 1$ or 2 or $3 \cdots$, or N and M_{rs} is the minor of a_{rs} (see Section R8.6).

Example R8.3. To calculate the determinant of the matrix

$$\mathbf{U} = \begin{bmatrix} 1 & 6 & -4 & 6 \\ -7 & 1 & 3 & -6 \\ 2 & -3 & 6 & 5 \\ -2 & 4 & 2 & 6 \end{bmatrix}, \tag{R8.10}$$

we first calculate the minors

$$M_{11} = \begin{bmatrix} 1 & 3 & -6 \\ -3 & 6 & 5 \\ 4 & 2 & 6 \end{bmatrix} = 320, \ M_{12} = \begin{bmatrix} -7 & 3 & -6 \\ 2 & 6 & 5 \\ -2 & 2 & 6 \end{bmatrix} = -344$$
$$M_{13} = \begin{bmatrix} -7 & 1 & -6 \\ 2 & -3 & 5 \\ -2 & 4 & 6 \end{bmatrix} = 232, \ M_{14} = \begin{bmatrix} -7 & 1 & 3 \\ 2 & -3 & 6 \\ -2 & 4 & 2 \end{bmatrix} = 200.$$

The determinant is therefore given by

$$\begin{aligned} \det(\mathbf{U}) &= 1 \cdot (-1)^{1+1} \cdot 320 + 6 \cdot (-1)^{1+2} \cdot (-344) + (-4) \cdot (-1)^{1+3} \cdot 232 + 6 \cdot (-1)^{1+4} \cdot 200 \\ &= 320 + 2064 - 928 - 1200 = 256. \end{aligned}$$

R8.6 Minor and Cofactor

From $\mathbf{U}$ given in Eq.(R8.7), the minor M_{rs} of a_{rs} in $\mathbf{U}$ is defined to be the determinant of the $(N-1) \times (N-1)$ matrix formed by deleting the r-th row and s-th column of $\mathbf{U}$. For example,

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2N} \\ a_{32} & a_{33} & \cdots & a_{3N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N2} & a_{N3} & \cdots & a_{NN} \end{vmatrix}, \ M_{12} = \begin{vmatrix} a_{21} & a_{23} & \cdots & a_{2N} \\ a_{31} & a_{33} & \cdots & a_{3N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N3} & \cdots & a_{NN} \end{vmatrix}.$$

The cofactor C_{rs} of a_{rs} in $\mathbf{U}$ given in Eq.(R8.7) is defined to be

$$C_{rs} = (-1)^{r+s} M_{rs}. \tag{R8.11}$$

R8.7 Inverse of a Matrix

By Eq.(R8.7), if $\det(\mathbf{U}) \neq 0$, then the inverse of $\mathbf{U}$ exists and is uniquely given by

$$\mathbf{U}^{-1} = \frac{1}{\det(\mathbf{U})} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{N1} \\ C_{12} & C_{22} & \cdots & C_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1N} & C_{2N} & \cdots & C_{NN} \end{bmatrix}, \tag{R8.12}$$

where $C_{rs} = (-1)^{r+s} M_{rs}$ is the cofactor of a_{rs} in $\mathbf{U}$ given in Eq.(R8.7).

Example R8.4. In this example we want to find the inverse of the matrix given in Eq.(R8.10). The cofactors are calculated as follows:

$$\begin{aligned}
&C_{11} = (-1)^{1+1} M_{11} = 320, && C_{12} = (-1)^{1+2} M_{12} = 344, \\
&C_{13} = (-1)^{1+3} M_{13} = 232, && C_{14} = (-1)^{1+4} M_{14} = -200 \\
&C_{21} = (-1)^{2+1} M_{21} = 176, && C_{22} = (-1)^{2+2} M_{22} = 210, \\
&C_{23} = (-1)^{2+3} M_{23} = 158, && C_{24} = (-1)^{2+4} M_{24} = -134 \\
&C_{31} = (-1)^{3+1} M_{31} = 240, && C_{32} = (-1)^{3+2} M_{32} = 234, \\
&C_{33} = (-1)^{3+3} M_{33} = 198, && C_{34} = (-1)^{3+4} M_{24} = -142 \\
&C_{41} = (-1)^{4+1} M_{41} = -344, && C_{42} = (-1)^{4+2} M_{42} = -329, \\
&C_{43} = (-1)^{4+3} M_{43} = -239, && C_{44} = (-1)^{4+4} M_{44} = 227.
\end{aligned}$$

Therefore, the inverse is

$$\mathbf{U}^{-1} = \frac{1}{256} \begin{bmatrix} 320 & 176 & 240 & -344 \\ 344 & 210 & 234 & -329 \\ 232 & 158 & 198 & -239 \\ -200 & -134 & -142 & 227 \end{bmatrix}.$$

R8.8 Unitary Matrix and Orthogonal Matrix

The $N \times N$ matrix $\mathbf{U}$ is said to be unitary if

$$\mathbf{U}^H \mathbf{U} = \mathbf{U}\mathbf{U}^H = kI, \tag{R8.13}$$

where k is any nonzero constant and $\mathbf{U}^H = (\mathbf{U}^T)^*$ is the conjugate-transpose of $\mathbf{U}$. Note that the unitary matrix is always invertible and $\mathbf{U}^H = \mathbf{U}^{-1}$.

A real unitary matrix $\mathbf{U}$ is also called an orthogonal matrix, i.e.,

$$\mathbf{U}^T\mathbf{U} = \mathbf{U}\mathbf{U}^T = kI, \tag{R8.14}$$

where k is any nonzero constant and $\mathbf{U}^T$ is the transpose of $\mathbf{U}$. Similarly, the orthogonal matrix is always invertible and $\mathbf{U}^H = \mathbf{U}^{-1}$. If $k = 1$, then the matrix $\mathbf{U}$ is said to be orthonormal.

R8.9 Cramer's Rule

Consider the set of N linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1N}x_N &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2N}x_N &= b_2, \\ &\vdots \\ a_{N1}x_1 + a_{N2}x_2 + \cdots\cdots + a_{NN}x_N &= b_N, \end{aligned} \tag{R8.15}$$

writing in matrix form yields

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix}. \tag{R8.16}$$

Let D be the determinant of the coefficient matrix

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{vmatrix}. \tag{R8.17}$$

If $D \neq 0$, then the system(R8.15) has the unique solution

$$x_1 = \frac{D_1}{D},\ x_2 = \frac{D_2}{D},\ \cdots,\ x_N = \frac{D_N}{D}, \tag{R8.18}$$

where

$$D_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1N} \\ b_2 & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ b_N & a_{N2} & \cdots & a_{NN} \end{vmatrix}, D_2 = \begin{vmatrix} a_{11} & b_1 & \cdots & a_{1N} \\ a_{21} & b_2 & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & b_N & \cdots & a_{NN} \end{vmatrix}, \cdots, \text{etc.}$$