Channel Equalizer Design Based on Wiener Filter and Least Mean Square Algorithms

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***Abstract*— This paper investigates the Wiener and least mean square (LMS) algorithms in the design of traversal tap delay line filters for the purpose of compensating the effect of the communication channel. The designed equalizers remove the distortion caused by the channel from the transmitted signal without requiring any specific model or state-space information. The first approach is based on the a recursive Wiener filtering procedure and is designed using the Wiener-Hopf equation. On the other hand, the second approach uses the LMS algorithm and investigates the effect of different step sizes on the speed of the conversion and the accuracy of the overall algorithm. Simulation results are presented and both schemes are compared under different distortion levels and signal to noise ratio(SNR) values via impulse response, frequency response and ABER simulations.**

***Index Terms*— Wiener-Hopf, least mean square, transversal tap delay line filters, and channel equalization.**

I. Introduction

ACCURATE estimation of the communication channel greatly affects the performance of communication systems operating over the medium [1], [2]. The signal transmitted over a channel, such as the fading channel, is affected by many distortions that result in both amplitude and phase fluctuations. Furthermore, the delay spread of the channel introduces inter symbol interference (ISI) to the received signal, which is one of the major obstacles to reliable and high-speed data transmission. Channel equalization is the process of compensating for the negative effect of the channel on the transmitted signal and removing the resulting ISI. To achieve this goal the equalizer uses an estimate of the channel frequency response, however the fading channel varies throughout the transmission cycle, requiring the equalizer to learn the frequency response in an adaptive fashion to be able to continuously mitigate the negative effect of the channel. In general, the training of the equalizer is done by sending a fixed-length known bit sequence over the channel and using the training sequence to determine the optimum tap weights for the equalizer [1].

Equalization techniques fall into two broad categories: linear and nonlinear. The linear techniques are generally the simplest to implement and to understand conceptually. However, linear equalization techniques typically suffer from noise enhancement, and are therefore not used in most wireless applications. Among nonlinear equalization techniques, decision-feedback equalization (DFE) is the most common, since it is fairly simple to implement and does not suffer from noise enhancement. The optimal equalization technique to use is maximum likelihood sequence estimation (MLSE). Unfortunately, the complexity of this technique grows exponentially with memory length, and is therefore impractical on most channels of interest. Tap updating algorithms, more specifically the proposed algorithm in [3] are also a popular method for equalization due to the lower complexity and the fact that they do not require any model or state-space formulation to extract the transmitted signal.

In this paper we investigate the use of adaptive Wiener filters and LMS algorithm in the design of channel equalizers. In the first scheme the adaptive algorithm represented in Fig. 1 is designed based on the Wiener-Hopf equations [4]. In the second approach the LMS algorithm [4] with different step sizes is applied to the design of the channel equalizer. Both scheme are compared in terms of performance for different signal to noise ratio (SNR) values and distortion levels. More specifically in the case of LMS algorithm the mean square error (MSE) is investigated for different levels of distortions and different step sizes. Throughout the document frequency and impulse response analysis is provided to support the concluding results.

This paper is organized as follows: Section II outlines the system model and establishes the algorithms under consideration to determine the optimum filter coefficients with respect to a minimum mean square (MSE) design criteria for both the Wiener and LMS algorithm. Section III discusses the extensive simulation results and examines both filter design algorithms in terms of their effectiveness in removing the disturbances.

This following notation is used throughout this report: italic letters (x) represent scalar quantities, bold lower case letters ($\mathbf{x}$) represent vectors, bold upper case letters ($\mathbf{X}$) represent matrices, and $(.)^T$ denotes transpose.

II. System and Filter Model

In this section we define the system model for the proposed project. Eq. (1) defines the relationship between the received signal $r(n)$ and the desired signal $d(n)$.

$$r(n) = h(n)d(n) + \nu(n), \tag{1}$$

where $h(n)$ represents the channel impulse response and $\nu(n)$ is the additive white Gaussian noise with mean zero and variance $\sigma_n^2 = .001$. Furthermore, $h(n)$ is represented as

$$h(n) = \begin{cases} .5\left[1+\cos\left(\frac{2\pi}{\xi}(n-2)\right)\right] & n = 1,2,3 \\ 0 & \text{otherwise} \end{cases}, \tag{2}$$

where ξ is the distortion parameter that controls the amplitude of the disturbance caused by the channel (the channel distortion is increased as ξ increases).

Two different approaches in the design of the channel equalizer are taken into consideration and they are the followings:

1) Design the adaptive equalizer using the Winer-Hopf Equation filter.
2) Design the adaptive Equalizer using the LMS algorithm.

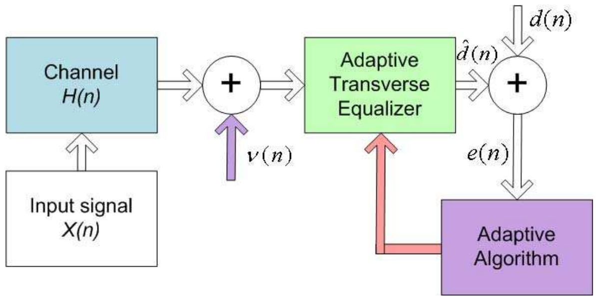

Fig. 1. The block diagram representing the two filter designs used to remove the disturbances from the received signal.

A. Equalizer Design Based on the Winer-Hopf Algorithm

An FIR filter, illustrated in Fig. 2, is used to estimate the channel, $h(n)$. Based on the design criteria the input and output relationship for the filter can be illustrated as

$$\hat{d}(n) = \sum_{k=0}^{M} w_k r(n-k), \tag{3}$$

where $\{w_0, w_1, \cdots, w_M\}$ represent the filter coefficients and M is the order of the FIR filter. Eq. (3) in vector form is represented as

$$\hat{d}(n) = \mathbf{w}^T \mathbf{r}, \tag{4}$$

where $\mathbf{w}^T$ is transpose of the $M \times 1$ vector of filter coefficients and $\mathbf{r}$ is the $M \times 1$ vector of the input parameters ($\mathbf{r} = \{r(n), r(n-1), \cdots, r(n-M)\}^T$. In general, $d(n)$ is not supposed to be known. However, in order to design the filter and to determine the optimal values of its tap weights, a short sequence of $d(n)$ must be made available. Based on the above system model the mean square error (MSE) for the above estimation problem can be defined as

$$\begin{aligned} j(n) &= E\left[(d(n) - \hat{d}(n))^2\right] \\ &= E[(d(n) - \mathbf{w}^T\mathbf{r})(d(n) - \mathbf{r}^T\mathbf{w})], \end{aligned} \tag{5}$$

where $j(n)$ is defined as the cost function. $j(n)$ can be rewritten as

$$j(n) = E\left[(d^2(n)] - 2\mathbf{w}^T E[\mathbf{r}d(n)] + \mathbf{w}^T E[\mathbf{r}\mathbf{r}^T]\mathbf{w}\right]. \tag{6}$$

Assuming that the input and the desired sequence are stationary zero-mean random processes, Eq. (6) can be modified as follows

$$j(n) = \sigma_d^2 - 2\mathbf{w}^T\mathbf{p} + \mathbf{w}^T\mathbf{R}\mathbf{w}, \tag{7}$$

where σ_d^2 is the variance of $d(n)$, $\mathbf{p}$ is the cross correlation vector between the input sequence and the desired sequence and is expressed as

$$\mathbf{p} = E[\mathbf{r}d(n)] = \begin{pmatrix} E[r(n)d(n)] \\ E[r(n-1)d(n)] \\ E[r(n-2)d(n)] \\ \vdots \\ E[r(n-M)d(n)] \end{pmatrix}, \tag{8}$$

and matrix $\mathbf{R}$ is the autocorrelation matrix of the input sequence and is defined in Eq. (9). The objective of this design is to determine the filter coefficients, $\mathbf{w}$, such that the cost function expressed in Eq. (7) is minimized.

In Eq. (7), the cost function is a quadratic function of $\mathbf{w}$ and can be minimized by taking its gradient with respect to $\mathbf{w}$ and setting the results to zero. The gradient of $j(n)|_{\mathbf{w}}$ is represents as

$$\bigtriangledown|_{\mathbf{w}}\mathbf{j}(n) = -\mathbf{p} + 2\mathbf{R}\mathbf{w} = 0. \tag{10}$$

Taking the second gradient of $\mathbf{j}$(n) in Eq. (10) with respect to $\mathbf{w}$ results in the Hessian matrix $\mathbf{H}$,

$$\bigtriangledown^2|_{\mathbf{w}}\mathbf{j}(n) = \mathbf{H} = 2\mathbf{R}. \tag{11}$$

where $h_{i,j}$ the ith row and jth column element of $\mathbf{H}$ is defined as

$$h_{i,j} = \frac{\partial^2 j(n)}{\partial w_i \partial w_j}. \tag{12}$$

Since the input sequence $r(n)$ is stationary, then the autocorrelation matrix, $\mathbf{R}$, is symmetric and positive semi-definite. This means that the Hessian matrix is positive semi-definite as well and consequently the solution of Eq. 10 leads to a minimum value for the cost function $j(n)$. Based on the results presented here the solution for the optimum filter tap weights is

$$\mathbf{w}_o = \mathbf{R}^{-1}\mathbf{p} \tag{13}$$

The results presented in Eq. (13) is known as the Wiener-Hopf Equation.

$$\mathbf{R} = E\left[\mathbf{r}\mathbf{r}^T\right] = \begin{pmatrix} E[r^2(n)] & E[r(n)r(n-1)] & \cdots & E[r(n)r(n-M)] \\ E[r(n-1)r(n)] & E[r^2(n-1)] & \cdots & E[r(n-1)r(n-M)] \\ \vdots & \vdots & \vdots & \vdots \\ E[r(n-M)r(n)] & E[r(n-M)r(n-1)] & \cdots & E[r^2(n-M)] \end{pmatrix} \tag{9}$$

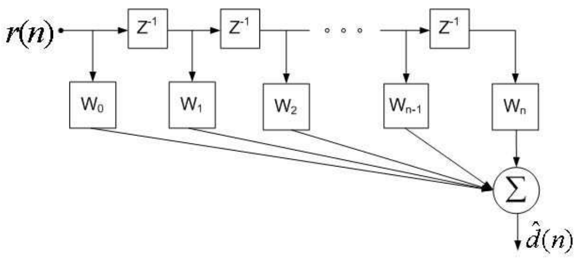

Fig. 2. The finite impulse response used to estimated $\hat{d}(n)$.

B. Equalizer Design Based on the LMS Algorithm

This section briefly discusses the formulation of the LMS algorithm. The cost function $j(n)$ which has been derived based on the MSE in its quadratic form can be presented as

$$j(n) = j_{\text{min}} + (\mathbf{w} - \mathbf{w}_o)^T \mathbf{R}(\mathbf{w} - \mathbf{w}_o), \tag{14}$$

where j_{min} represents the minimum mean square error (MMSE) corresponding to the optimal filter weights, $\mathbf{w}_o$. By taking the gradient of $j(n)$ in Eq. (14) with respect to the filter weights and moving in small steps in the opposite direction of the gradient vector, the following relationship between the filter coefficients can be found as

$$\mathbf{w}_{n+1} = \mathbf{w}_n - \frac{\mu}{2} \bigtriangledown j(n)|_{\mathbf{w}(n)}, \tag{15}$$

where the negative sign guarantees that the movement is in the negative direction of the gradient and the parameter μ is the step size. The choice of μ dictates the convergence speed of the algorithm and also the value of the MMSE. The smaller the value of μ the lower the MMSE, however the slower the algorithm converges to the optimum filter weights. After further algebraic manipulation Eq. (15) can be represented as [4]

$$\mathbf{w}_{n+1} = \mathbf{w}_n + \frac{\mu}{2} e(n)\mathbf{r}(n), \tag{16}$$

where $e(n)$ is the error function, defined as $d(n) - \hat{d}(n)$ and can be further expressed as

$$e(n) = d(n) - \mathbf{w}^T(n)\mathbf{r}(n). \tag{17}$$

III. Simulation Results

In this section simulation results based on the two filter designs are presented and investigated. The Wiener and LMS equalizers based on the system model presented in the previous section are developed and put to the test for different values of ξ, SNR, and step size in the case of LMS algorithm. The filter order is set to $M = 11$.

A. Adaptive Equalizer based on the Winer Filter Design

The Wiener filter design presented in Section II has been applied in the design of the equalizer. Fig. 3 represents the input output relationship for the filter. The data is generated based on a uniform distribution.

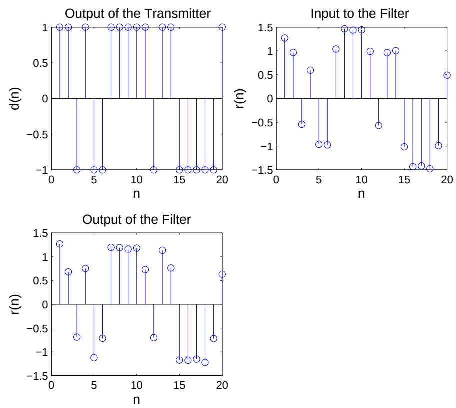

Fig. 3. The output of the transmitter, the input to the filter, and the output of the Wiener equalizer.

Fig. 4 represents the impulse response of Wiener equalizer. The impulse response of the filter is not symmetric, therefore the filter does not acquire linear phase catechistical. This complicates the performance of the equalizer, since a nonlinear phase filter sufferers from variable group delay, thus different frequency components experience different delay times. Consequently, the resulting output is distorted, diminishing the performance of the digital receiver.

Fig. 5 and Fig. 6 represented the magnitude and phase response of the Wiener equalizer. The results in Fig. 5 demonstrate that as deduced from Fig. 4 the Wiener equalizer is not a linear phase filter. From the results illustrated in Fig. 6 it is clear that the wiener equalizer is a bandpass filter and the response of the filter seeks to eliminate the distortion caused by the channel. However, as illustrated in Fig. 6, the filter is not capable of eliminating the distortive effect of the channel completely. This is one of the major drawback of this Wiener equalizer and demonstrated in

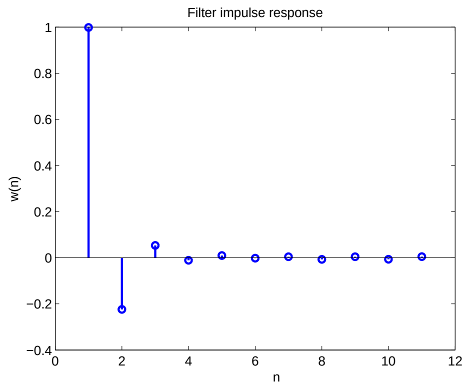

Fig. 4. The impulse response of the Wiener equalizer with $\xi = 2.9$ and $\sigma_n^2 = .001$.

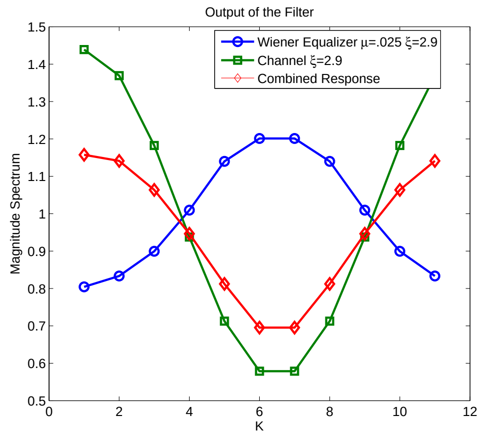

Fig. 6. The magnitude spectrum of the Wiener equalizer, the channel, and the combined system.

ABER analysis presented below the performance of the Wiener greatly diminishes as ξ increases.

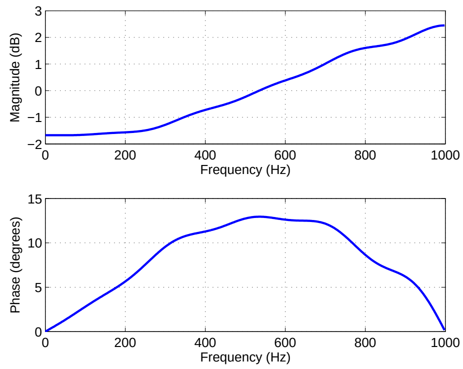

Fig. 5. The magnitude and phase response of the Wiener equalizer.

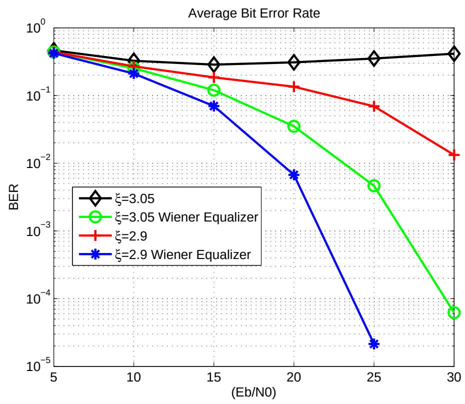

Fig. 7. ABER plots for digital system equipped with and without the Wiener equalizer at $\xi = 2.9$ and $\xi = 3.05$ distortion levels.

Fig. 7 represents the average bit error rate (ABER) analysis for the system with and without the Wiener equalizer for $\xi = 2.9$ and $\xi = 3.05$. The SNR values are set to $\{5, 10, 15, 20, 25, 30\}$. The results presented in Fig. 7 demonstrate the performance advantage of a communication system equipped with the Wiener equalizer. At $\xi = 2.9$ distortion levels the Wiener equalizer provides a 10dB performance gain compared to a system without the equalizer, which is a significant improvement. At $\xi = 3.05$ it is clear from the results that a communication system without the equalizer fails and is not capable of receiving the transmitted signal correctly. However, the system equipped with the equalizer significantly reduces the effect of the distortion and reaches an ABER of 10^{-4} at SNR=30dB, demonstrating that the system is applicable in a practical communication system.

B. Adaptive Equalizer based on the LMS Algorithm

This section presents the results for the LMS equalizer design with the filter order the same as the case of the Wiener equalizer. In this section the effect of different distortion levels, SNR values, and step sizes on the performance of the LMS equalizer are been investigated and presented. (the effect of step size on the achievable minimum mean square error (MMSE) of the LMS filter is discussed).

Fig. 8 represents the input output relationship for the LMS equalizer. As noted in Fig. 8, the digital input to the filter suffers from considerable distortion caused by the channel and the additive noise. However, the output of the LMS equalizer is capable of removing a significant portion of the distortions and compared to the Wiener equalizer, the LMS equalizer performs considerably better. It is important to note that the output of the LMS equalizer is delayed by 7 samples due to the delay of the channel and the filter itself (delay=$\lfloor$channel order$/2\rfloor + \lfloor$filter order$/2\rfloor$).

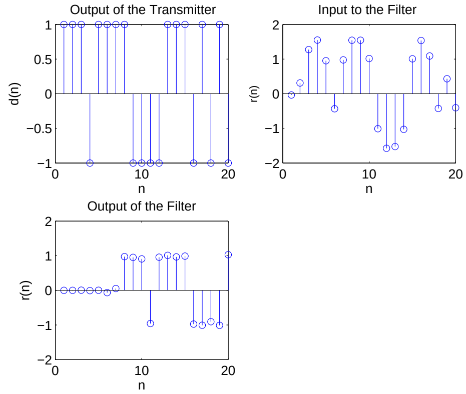

Fig. 8. The output of the transmitter, the input to the filter, and the output of the filter for the LMS equalizer.

Fig. 9 illustrates the impulse response of the LMS equalizer. Unlike the Wiener equalizer, the LMS filter has a symmetric impulse response and is linear phase. linear phase filters have a constant group delay, where all frequency components have equal delay times, resulting in no distortion due to frequency selectivity. This is a desired property in the design of equalizers.

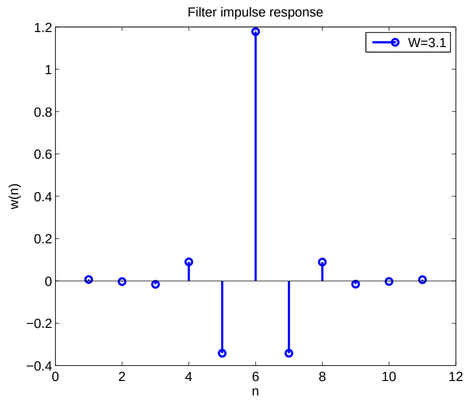

Fig. 9. The impulse response of the LMS equalizer with $\xi = 2.9$ and $\sigma_n^2 = .001$.

Fig. 10 and Fig. 11 demonstrate the magnitude and phase response of the LMS equalizer. As deduced from the results in Fig. 9 the filter acquires linear phase characteristics. Fig. 11 illustrates the magnitude spectrum of the LMS equalizer and the channel. Similar to the Wiener filter the equalizer is a bandpass filter, however the spectrum of the LMS equalizer is capable of completely eliminating the effect of the channel distortion at $\xi = 2.9$. This result clearly demonstrates that the LMS equalizer outperforms the Wiener algorithm. The magnitude spectrum of LMS equalizer combined with its linear phase characteristics, let us to conclude that the LMS equalizer is a superior filter compared to the Wiener equalizer.

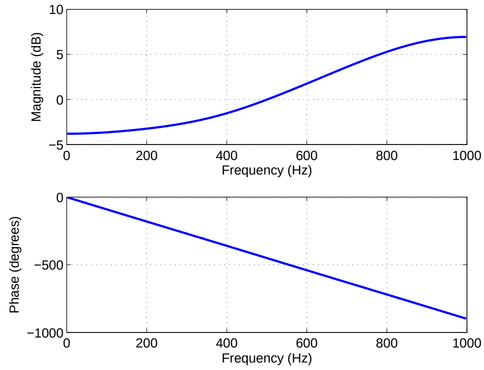

Fig. 10. The magnitude and phase response of the LMS equalizer.

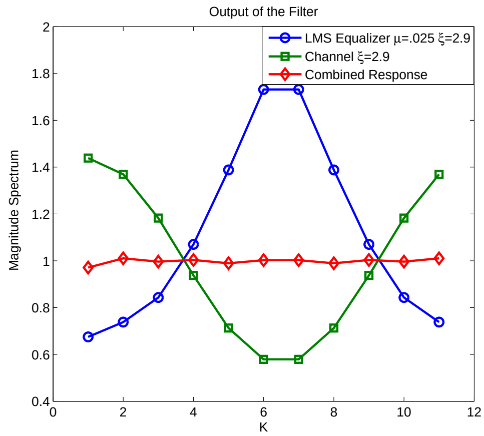

Fig. 11. The magnitude spectrum of the LMS equalizer, the channel, and the combined system.

One of the main goals of this project is to investigate the effect of the step size μ on the MSE of the LMS algorithm. As stated previously a larger step size results in a faster convergence of the algorithm, however it also results in a larger MMSE. Fig. 12 represents the MSE derived in Eq. (17) for different values of μ. At $\mu = .075$ the algorithm tends to converge to the optimal filter coefficients w_o in 100 iterations and the MMSE is approximately 6×10^{-3}. At $\mu = .025$ the convergence takes 500 iterations, however the resulting MMSE is approximately 2×10^{-3}. Thus, if the desired communication system can tolerate the added delay, $\mu = .025$ would be a better choice. At $\mu = .0075$ the system does not converge after 1500 iterations and it requires even more delay to reach w_o. Thus, this step size is deemed too small since it does not converge in a acceptable time cycle and requires significantly more delay, resulting in lower throughput. Fig. 12 demonstrates that the step size

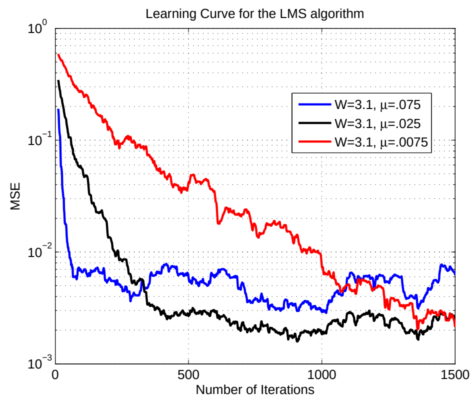

Fig. 12. MSE for the LMS equalizer at $\mu = .075$, $\mu = .025$, and $\mu = .0075$.

parameter, μ is an important design criteria and needs to be chosen appropriately to reach the desired MMSE at a reasonable delay.

Fig. 13 represents the ABER performance of the LMS equalizer for different step sizes at different distortion levels. The performance of the system at $xi = 3.35$ drops off quite significantly. This is expected since the equalizer is only capable of removing the distortive effect of the channel to a certain degree. The performance difference at $\mu = .1$ $\mu = .01$ is approximately 2dB which demonstrates the importance of the step size in the performance of the communication system (3dB represents the performance difference between a coherent and non-coherent communication system).

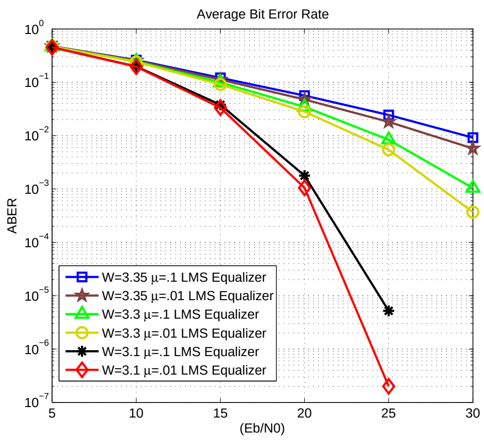

Fig. 13. The desired signal and input signal presented in a span of 1 second.

Fig. 14 provides a performance comparison between the Wiener equalizer and the LMS equalizer in terms of ABER. As expected based on the phase and magnitude analysis the LMS algorithm performs better and the difference between the two systems is quite significant. The LMS algorithm outperforms the Wiener equalizer by 5dB. From a communication stand point this is a considerable performance gap between the two systems and it renders the Wiener filter design very inefficient in comparison to the LMS equalizer.

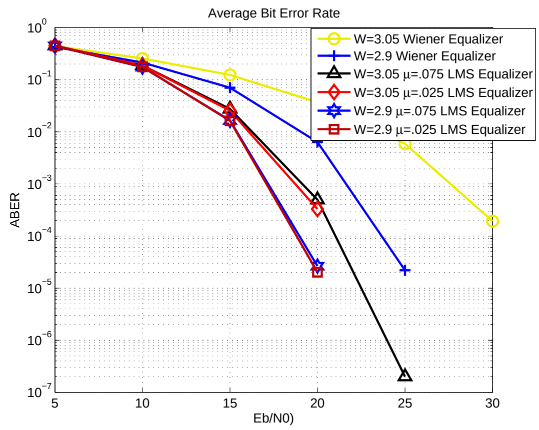

Fig. 14. The desired signal and input signal presented in a span of 1 second.

IV. Conclusion

In this paper channel estimation using both the LMS and Wiener algorithm were presented and investigated. A digital signal traveling through the transmission medium is distorted and the goal of the equalizer is to eliminate the effect of channel distortions, resulting in significant performance gains. In the first approach the Wiener-Hopf equation was applied to determine the equalizer filter coefficients based on a training sequence. The second approach involved the use of the LMS algorithm to determine the optimal set of filter coefficients that minimize the MSE. Both schemes were compared based on their magnitude and phase response and it was clearly demonstrated that due to its linear phase characteristics and superior magnitude spectrum, the LMS equalizer is the better choice. The ABER investigations and simulations demonstrated that the LMS algorithm outperforms the Wiener equalizer by 5dB which is a very significant performance gap. Finally, we investigated the effect of the step size parameter on the convergence speed and MMSE performance of the LMS algorithm, and it was demonstrated that step size has a significant roll in the performance of LMS equalizer and is an important design criteria.

References

[1] Andrea Goldsmith, *Wireless Communications*. Cambridge University Press, 2004.

[2] Bernard Sklar, *Digital Communications*. Pearson Education Press, 2005.

[3] J.G. Proakis, “Adaptive equalization for tdma digital mobile radio,” *IEEE Trans. Vehic. Technol.*, vol. 40, no. 2, pp. 333–341, 1991.
[4] B. Widrow and S. D. Stearns, *Adaptive Signal Processing*. Prentice Hall Signal Processing Series, 1985.