ARMA Synthesis of Fading Channels

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Abstract

Computationally scalable and accurate estimation, prediction, and simulation of wireless communication channels is critical to the development of more adaptive transceiver algorithms. Previously, the application of autoregressive moving average (ARMA) modeling to fading processes has been complicated by ill-conditioning and nonlinear parameter estimation. This correspondence presents a numerically stable and accurate method to synthesize ARMA rational approximations of correlated Rayleigh fading processes from more complex higher order representations. Here, the problem is decomposed into autoregressive (AR) model matching followed by linear system identification. Performance is compared to that of AR, inverse discrete Fourier transform, and sum of sinusoids techniques. Also, for the first time, the finite-precision performance of different methods is compared.

Index Terms

Autoregressive Moving Average (ARMA), Autoregressive (AR), Moving Average (MA), Inverse Discrete Fourier Transform (IDFT), Sum of Sinusoids (SOS), finite numerical precision.

I. Introduction

The properties of a mobile fading channel significantly influence the design of wireless devices. This has motivated extensive research into the statistical modeling of Rayleigh fading channels, which is also a core component of more complex scattering models. Clarke's fading model [1], or a simplified version proposed by Jakes [2], have been widely used for simulation. There also exist a variety of implementations of these models, ranging from the sum of sinusoids (SOS) [3]- [5], IDFT [6]- [7], AR [8], and ARMA schemes [9]- [11]. The limitations of SOS are outlined in [5] and were addressed, to some extent, in [3]. However the SOS model

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fundamentally requires the summation of numerous sinusoids to generate Rayleigh variates with the correct statistics. The IDFT method, can offer improved accuracy at a cost of requiring an inverse fast Fourier transform (IFFT) on a large block of samples ($N = 2^{15}$ or larger). Performing the IFFT operation on such a large number of samples, however, results in large delay and overhead and may be computationally impractical for generating shorter sequences with different Doppler parameters, which is important for modeling more complicated mobile fading dynamics. In contrast, ARMA modeling has the potentially similar accuracy to a large size IDFT, with significantly fewer computations. It is important to note that high-order AR systems are required for accurate approximation to the non-rational Jakes' frequency spectrum. These high-order AR systems can, in principle, be approximated with considerably lower order ARMA filters. Previous ARMA-based methods proposed in [9]- [11], determine poles and zeros separately, and as a result, are still of very high order, typically ranging from 200-1000.

In the following, a low-order ARMA synthesis technique is developed that can generate high quality Rayleigh variates. The resulting ARMA system could then be applied to system design and performance assessment in areas such as power control for broadband and CDMA systems [12]- [13], channel estimation using Kalman filters [14]- [16], and blind detection and decoding [17]- [20]. In addition to computational considerations, this correspondence compares the finite-precision performances of the above fading channel simulation techniques, filling a gap in the previous literature.

II. ARMA Model Generation

An ARMA(p, q) model of p poles and q zeros has a potential to generate digital filters with closely matching second-order statistics. An equivalent AR(P) model of order P would require $P >> p + q$ [21]. Unfortunately, the estimation of the ARMA model parameters leads to a nonlinear least squares optimization problem. Therefore, suboptimal schemes, with reduced computational complexity have been proposed in the literature, e.g., [21]- [23], that estimate the p denominator and q numerator parameters separately. However, these decoupled estimation algorithms result in ARMA systems with high order.

The relationship between the autocorrelation function $r_{xx}[m]$ and ARMA(p, q) parameters is given by [24], [25]:

$$r_{xx}[m]\begin{cases} r_{xx}^*[-m] & m<0 \\ -\sum_{k=1}^{p} a[k]r_{xx}[m-k]+ & \\ \sigma_w^2 \sum_{k=m}^{q} b[k]h[k-m] & 0\le m\le q \\ -\sum_{k=1}^{p} a[k]r_{xx}[m-k] & m>q \end{cases} \tag{1}$$

where $r_{xx}[m], -\infty < m < \infty$ is the desired autocorrelation sequence of the fading process, $b[k], 0 \le k \le q$ and $a[k], 0 \le k \le p$ represent the coefficients of the numerator and denominator polynomials of the ARMA transfer function, respectively, $h[m], 0 \le m < \infty$ is the corresponding time-domain impulse response sequence, and σ_w^2 is the variance of the input driving sequence. Attempting to determine the ARMA parameters by solving Eq. (1) results in a non-linear set of equations, because the impulse response is also a function of the unknown ARMA parameters.

Suboptimal methods that simultaneously estimate all the $a[k]$ and $b[k]$ parameters are presented in [26], [27]. However due to the fact that the autocorrelation sequence under consideration is a narrowband process and is not rational [2], none of the above schemes reliably result in a stable ARMA filter.

The proposed solution first employs a high-order AR approximation to synthesize a rational model. Note that the AR(P) model is able to match the first P autocorrelation lags of any wide sense stationary random process exactly.

For an order-P AR process, Eq. (1) simplifies to

$$r_{xx}[m]\begin{cases} r_{xx}^*[-m] & m<0 \\ -\sum_{k=1}^{P} ar[k]r_{xx}[m-k]+\sigma_w^2 & m=0 \\ -\sum_{k=1}^{P} ar[k]r_{xx}[m-k] & m>0. \end{cases} \tag{2}$$

This gives rise to the following Yule-Walker equations, which when solved yield $ar[k], 1 \le k \le P$, the parameters of the AR(P) filter:

$$\begin{bmatrix} r_{xx}[0] & r_{xx}[-1] & r_{xx}[-2] & \dots & r_{xx}[-P+1] \\ r_{xx}[1] & r_{xx}[0] & r_{xx}[-1] & \dots & r_{xx}[-P+2] \\ . & . & . & & . \\ . & . & . & & . \\ . & . & . & & . \\ r_{xx}[P] & r_{xx}[P-1] & r_{xx}[P-2] & \dots & r_{xx}[0] \end{bmatrix} \begin{bmatrix} ar[1] \\ ar[2] \\ . \\ . \\ . \\ ar[P] \end{bmatrix} = - \begin{bmatrix} r_{xx}[0] \\ r_{xx}[1] \\ . \\ . \\ . \\ r_{xx}[P] \end{bmatrix} \tag{3}$$

This system of equations (3) can be efficiently solved using the Levinson-Durbin algorithm. Details on application to Rayleigh fading channels can be found in [8].

The resulting ARMA system is then determined by formulating a system identification problem [21], [22]. The input to the ARMA system consists of the sequence $x(n)$ that is generated by the AR(P) system driven by white noise $w[k] \sim WGN(0, \sigma_w^2)$ (see Figure 1). The input-output equation is given by

$$a_0 x(n) = -\sum_{k=1}^{p} a_k x(n-k) + \sum_{k=0}^{q} b_k w(n-k). \tag{4}$$

Setting $a_0 = b_0 = 1$, without loss of generality, Equation (4) can be expressed as

$$x[n] = \mathbf{z}^T(n-1)\mathbf{c}_{ARMA} + w(n) \tag{5}$$

where

$$\mathbf{z}[n] = [-x(n) \ldots - x(n-p+1) \;\; w(n) \ldots w(n-q+1)]^T \tag{6}$$

and the vector of filter coefficients,

$$\mathbf{c}_{ARMA} = [a[1] \ldots a[p] \;\; b[1] \ldots b[q]]^T. \tag{7}$$

Assuming that the excitation $w(n)$ is known we may predict $x(n)$ from past values, using the following linear predictor:

$$\hat{x}(n) = \mathbf{z}^T(n-1)\hat{\mathbf{c}}_{ARMA} \tag{8}$$

$$\hat{\mathbf{c}}_{ARMA} = \left[\hat{a}[1] \ldots \hat{a}[p] \;\; \hat{b}[1] \ldots \hat{b}[q]\right]. \tag{9}$$

The prediction error

$$e(n) = x(n) - \hat{x}(n) = x(n) - \mathbf{z}^T(n-1)\hat{\mathbf{c}}_{ARMA} \tag{10}$$

equals $w(n)$ if $\mathbf{c}_{ARMA} = \hat{\mathbf{c}}_{ARMA}$. Minimization of the total squared error

$$\xi(c) = \sum_{n=N_i}^{N_f} e^2(n) \tag{11}$$

leads to the system of linear equations

$$\hat{\mathbf{R}}_z \hat{\mathbf{c}}_{ARMA} = \hat{\mathbf{r}}_z \tag{12}$$

where the correlation of the output AR process

$$\hat{\mathbf{R}}_z = \sum_{n=N_i}^{N_f} \mathbf{z}(n-1)\mathbf{z}^T(n-1) \tag{13}$$

and the cross correlation

$$\hat{\mathbf{r}}_z = \sum_{n=N_i}^{N_f} \mathbf{z}(n-1)x(n). \tag{14}$$

Therefore a total of $p+q$ equations need to be solved to determine the parameters of the ARMA model. The use of residual windowing implies that $N_i = max(p,q)$ and $N_f = N-1$, where N, the number of inputs and outputs generated by the AR(P) filter, is chosen large enough to approximate Eqs. (13) and (14) by their expectations. It is important to note that the resulting ARMA filter is not guaranteed to be minimum phase and reflection of poles and zeros inside the z-plane unit-circle may be required. The resulting ARMA filter is of considerably lower order and therefore more numerically stable as will be described in the next section.

III. Numerical Results

Using the method outlined above, an AR(50) model ($P = 50$) was approximated by an ARMA(12,12) model. The normalized maximum doppler frequency, $f_m = .05Hz$ and $N = 2^{20}$ inputs and outputs were used to determine the parameters of the ARMA model. Since different Doppler values result in a horizontal-axis scaling of the Jakes frequency spectrum, it is sufficient to consider only one typical Doppler value in detail. Figures 2 and 3 represent the autocorrelation sequence of the Rayleigh variates generated using the example ARMA(12,12) digital filter. Comparing the second-order statistics of the variates generated using the ARMA(12,12) filter to that of the AR(50), it is clear that the ARMA(12, 12) filter closely matches Jakes autocorrelation sequence. Possible criteria for selecting the ARMA model order are discussed in [28] and may be applied here, but is beyond the scope of this paper. However, it is also noted in [28] that these criteria seem to work well only for a true AR or ARMA process. For $f_m = .05\ Hz$ and $N = 2^{20}$, we obtained impressive results with $p = 12$ and $q = 12$.

Next, using the quality measures described in [6], the quality of the variates $y(n)$ generated using the ARMA(12,12) filter are compared to that of the SOS [4], IDFT [6], and AR [8] models. The two quality measures are defined as follows. The first, termed *the mean power margin*, is defined by [6]

$$g_{mean} = \frac{1}{\sigma_y^2 L} trace\{\mathbf{C}_y \mathbf{C}_{\hat{y}}^{-1} \mathbf{C}_y\} \tag{15}$$

and the second, the *maximum power margin*, is defined by [6]

$$g_{max} = \frac{1}{\sigma_y^2} \max\{diag\{\mathbf{C}_y \mathbf{C}_{\hat{y}}^{-1} \mathbf{C}_y\}\} \tag{16}$$

where σ_{y^2} is the variance of the reference distribution. In (15) and (16), the $L \times L$ matrix $\mathbf{C}_{\hat{y}}$ is defined to be the covariance matrix of any length-L subset of adjacent variates produced by the random variate generator. Due to the stationarity of the generator output, the covariance matrix of all such subsets would be identical. The $L \times L$ covariance matrix of a reference vector of L ideally distributed variates is similarly defined to be $\mathbf{C}_y$. The matrix $\mathbf{C}_y$ represents the desired covariance matrix, and is known exactly (in this case the zeroth order bessel function). A value of $L = 200$ is chosen for consistency with the simulation results presented in [4], [6], and [8].

Table I summarizes the effects of the available numerical precision on the quality of the generated variates. Perfect variate generation corresponds to 0 dB for both measures implying zero deviation from the perfect autocorrelation sequence. The autocorrelation sequence of length 200 in [6], Eqs. (22) and (23) was considered for evaluation. The results presented in Table I demonstrate that the variate generating capability of the ARMA (12,12) filter compares favorably to that of the AR(50) filter in terms of quality and significantly outperforms the AR(20) filter. Table I also provides an AR (17,20) filter which is based on an AR (100) design prototype that outperforms the above AR(50) at lower complexity. It should be noted that since the ARMA(12,12) method is based on approximating an AR(50) model via system identification, the quality of its variates are bounded by that of the AR(50) model. It should also be mentioned that transient effects on ARMA filtering can be removed in a similar manner as described for the case of AR filters using the scheme proposed in [8].

In terms of computational requirements, it is worth noting from Table I that the 2^{15} length IDFT method has accuracy comparable to that of the ARMA(12,12) but requires about $2log_2(N) = 30$ real multiplications per unit time (MPUs) and a delay of 2^{15} samples. On the other hand, the ARMA(12,12) filter requires 24 real MPUs and a delay of only 12 samples. This is also clearly less than that of the AR(50) filter (50 MPUs and 50 samples delay). The superior performing ARMA (17,20) filter requires 37 MPUs and a 20-sample delay, significantly less than the computation required for its prototype AR(100) filter. Although determining the ARMA coefficients requires the added complexity of solving an extra $p+q$ equations, this only needs to be determined once offline.

More significantly, if a relatively small number of samples, say a few hundred, are needed at a particular Doppler value, the total computation using the IDFT approach cannot be scaled down without a significant loss in accuracy whereas the ARMA method is scalable.

IV. Finite Numerical Precision Effects

Due to the presence of poles near the unit circle in the Jakes spectrum, it is imperative to investigate the performance of the different Rayleigh variate generating schemes under various degrees of quantization. This aspect of the problem has not been addressed previously e.g., not in [4]- [8]. Table II compares the performance of the ARMA, AR, IDFT, and SOS methods under 28, 24, 22, 20, and 16 bits of quantization to represent varying degrees of precision. Equations (15) and (16) were used again to determine the quality of the generated variates. The quantization was applied to the inputs, outputs, filter coefficients, and the output of the IFFT operation. It is noteworthy that the AR and ARMA schemes tend to fail, i.e., result in unstable filters when the data is quantized below 22 bits. This is due to the fact that the poles of both sets of filters lie extremely close to the z-plane unit circle and therefore the stability of the system is quite sensitive to quantization. This latter effect exhibits the particular difficulties with the Jakes spectrum, which in theory, does not correspond to a positive definite autocorrelation spectrum and is therefore very ill-conditioned. On the other hand the SOS scheme, even though more computationally complex, tends to perform the best under limited numerical precision and therefore should be seriously considered for such applications.

V. Conclusion

By separating the issues of ill-conditioning and ARMA/AR equivalences, ARMA filters for the generation of Rayleigh random variates were considered. The inherent nonlinearity in determining the ARMA filter coefficients was addressed by first creating an AR approximation filter. The AR filter was then approximated by a significantly lower order ARMA filter through a linear system identification process. The complexity-performance tradeoff was found to be particularly attractive for generating small numbers of samples as needed in applications involving time-varying Doppler frequencies. Through a finite numerical precision study it was also determined that typically AR and ARMA schemes require a minimum of 22-bit quantization to result in stable filters.

REFERENCES

[1] R. H. Clarke, "A statistical theory of mobile-radio reception," *Bell Syst. Tech. J.*, pp. 957-1000, Jul.-Aug. 1968..

[2] W.C. Jakes, "Microwave Mobile Communications," *New York: John Wiley and Sons*, 1974.

[3] C. Xiao; Y. R. Yahong, N. C. Beaulieu, "Novel Sum-of-Sinusoids Simulation Models for Rayleigh and Rician Fading Channels," *IEEE Trans. on W. Communcations*, pp.3667-3679, vol. 5, no. 12, Dec. 2006.

[4] Y. R. Zheng & C. Xiao, "Simulation Models With Correct Statistical Properties for Rayleigh Fading Channels," *IEEE Trans. on Communcations*, pp.920-928, vol. 51, no. 6, June 2003.

[5] M. F. Pop & N. C. Beaulieu, "Limitations of sum-of-sinusoids fading channel simulators," *IEEE Trans. on Communcations*, IEEE Trans. Commun., vol. 49, pp. 699-708, Apr.2001.

[6] D. J. Young & N. C. Beaulieu, "The Generation of Correlated Rayleigh Random Variates by Inverse Discrete Fourier Transform," *IEEE Trans. on Communications*, pp.1114-1127, vol. 48, no. 7, July 2000.

[7] J. I. Smith, "TA computer generated multipath fading simulation for mobile radio," *IEEE Trans. Veh. Technol.*, vol. VT-24, pp. 3940, Aug. 1975.

[8] K. E. Baddour & N. C. Beaulieu, "Autoregressive Modeling for Fading Channel Simulation," *IEEE Trans. on Wireless Communications*, pp.1650-1662, vol. 4, no. 4, July 2005.

[9] W. Turin, R. Jana, C. Martin, & J. Winters, "Modeling wireless channel fading," *IEEE Vehicular Technology Conference*, pp.1740-1744, vol. 3, no. 4, Oct 2001.

[10] D. Schafhuber, G. Matz, & F. Hlawatsch, "Simulation of wideband mobile radio channels using subsampled ARMA models and multistage interpolation," *Proc. of IEEE Signal Proc. W.*, pp.571-574, Aug. of 2001.

[11] G. W. K. Colman , S. D. Blostein & N. C. Beaulieu, "An ARMA Multipath Fading Simulator," *Department of Electrical & Computer Engineering, Queen's University.*

[12] M. Sternad, T. Ekman, & A. Ahlh, "Power Prediction on Broadband Channels," *IEEE VTS 53rd on Vehicular Tech. Con.*, pp.2328-2332, vol.4, May 2001.

[13] B. Chen, B. Lee, & S. Chen, "Adaptive Power Control of Cellular CDMA Systems via the Optimal Predictive Model," *IEEE Trans. on W. Communications*, pp.1914-1927, vol. 4, no. 4, July 2005.

[14] K. S. Shanmugan, M. P. Sanchez, L. D. Haro, & M. Calvo, "Channel Estimation for 3G Wideband CDMA Systems Using the Kalman Filtering Algorithm," *IEEE Int. Con. on P. W. Comm.*, pp.95-97, Dec. 2000.

[15] M. J. Omidi, S. Pasupathy, & P. G. Gulak, "Joint Data and Kalman Estimation of Fading Channel Using A Generalized Viterbi Algorithm," *IEEE I. Conf. on Converging Techs.*, pp.1198-1203, vol.2, June 1996.

[16] J. Zhang & P. M. Djuric', "Joint Channel Estimation and Decoding of Space-Time Trellis Codes," *Proceedings of the 11th IEEE Signal Proc. Workshop*, pp.559-562, August 2001.

[17] M.E. Rollins & S.J. Simmons, "Reduced-search blind trellis decoders for frequency-selective Rayleigh fading channels," *IEEE Conference on Comm., Computers and S.P.*, pp.305-309, vol.1, May 1993.

[18] Y. Huang, J. Zhang, I.T. Luna, P.M. Djuric, &D.P.R. Padillo, "Adaptive blind multiuser detection over flat fast fading channels using particle filtering," *IEEE 2004 GLOBECOM*, pp.2419-2423 , vol.4, Nov 2004.

[19] M. Riediger & E. Shwedyk, "Communication receivers based on Markov models of the fading channel," *IEE Proceedings Communications*, pp.275-279 , no.4, vol.150, Aug 2003.

[20] S. Verdu, Brian D. 0. Anderson, & R. A. Kennedy, "Blind Equalization Without Gain Identification," *IEEE Trans. on Information Theory*, pp.292-297, vol. 39, no. 1, January 1993.

[21] D. G. Manolakis, V. K. Ingle, & S. M. Kogon, "Statitics and Adaptive Signal Processing," *McGraw-Hill Series*, 2000.

[22] Ljung L., "System Identification Theory For the User," *Prentice-Hall*, 1987.

[23] S. Li, Y. Zhu, & B. W. Dickinson "A Comparison of Two Linear Methods of Estimating the Parameters of ARMA Models," *IEEE Trans. on Automatic Control*, pp.915-917, August 1989.

[24] A. El-Jaroudi, J. Makhoul, "Digital Spectral Analysis With Applications," *IEEE Acoustics, Speech, and Signal Proc.*, pp.2162-2165, vol. 4, May 1989.

[25] S. L. Marple Jr., "Discrete pole-zero modeling and applications," *Prentice-Hall*, 1987.

[26] Steiglitz, K., & L. E. McBride, "A Technique for the Identification of Linear Systems," *IEEE Trans. on Automatic Control*, AC-10, pp.461-464, October 1965.

[27] Kay, S. M., "Modern Spectral Estimation," *Prentice-Hall*, 1987.

[28] C. W. Therrien, "Discrete Random Signals and Statistical Signal Processing," Englewood Cliffs (NJ), *Prentice-Hall*, 1992.

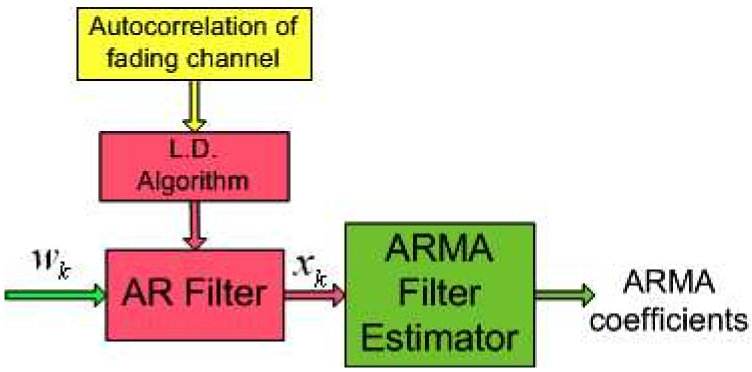

Fig. 1. Block diagram representing the information flow to determine the coefficients of the ARMA filter

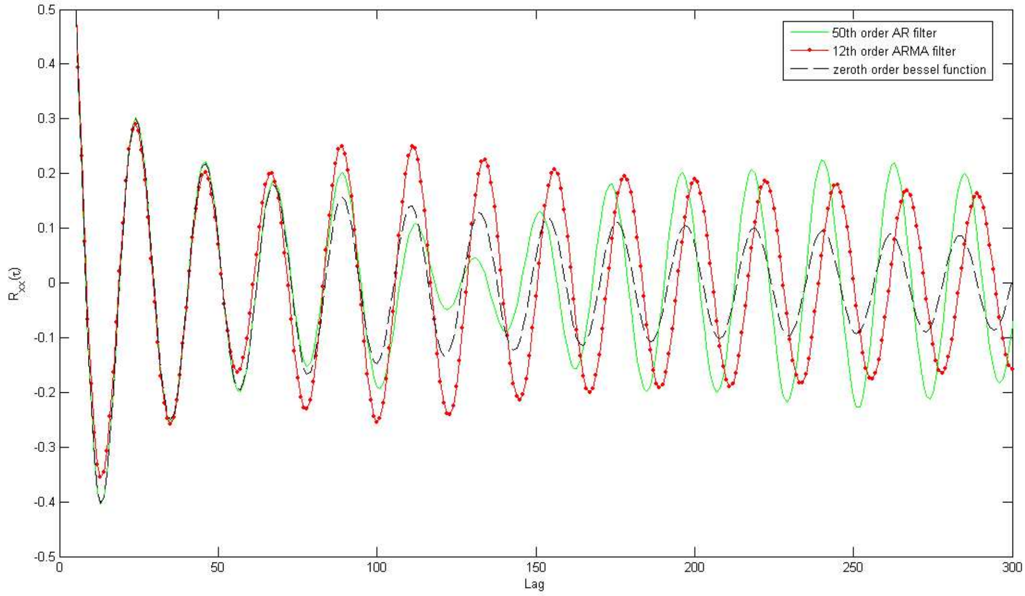

Fig. 2. Autocorrelation for ARMA(12) and AR(50) filters

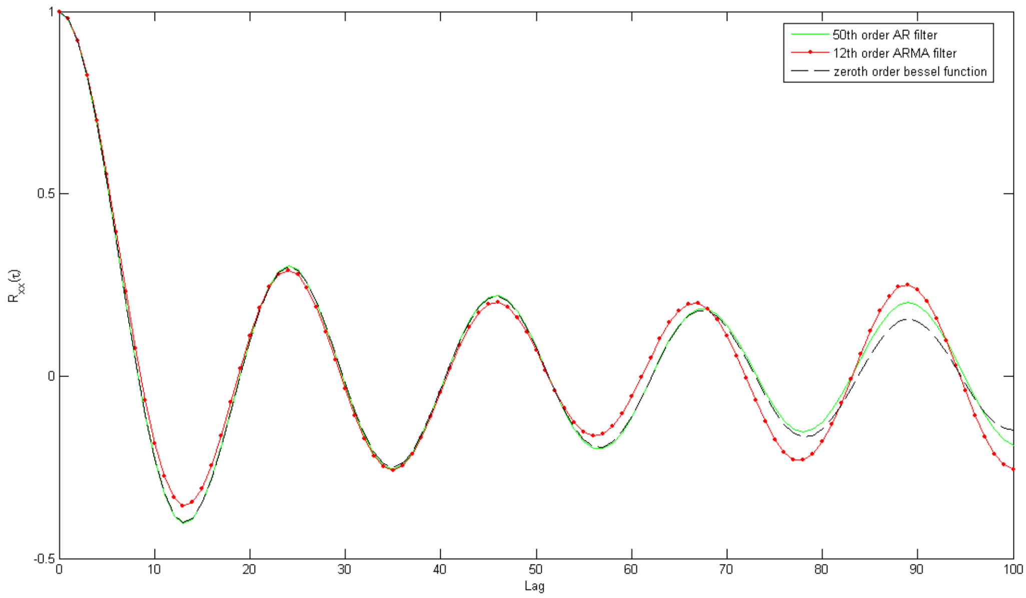

Fig. 3. Autocorrelation for ARMA(12) and AR(50) filters

TABLE I

A COMPARISON OF METHODS OF GENERATING BANDLIMITED RAYLEIGH VARIATES USING A COVARIANCE SEQUENCE OF LENGTH 200. PERFECT VARIATE GENERATION CORRESPONDS TO 0 DB FOR BOTH MEASURES.

	g_{mean}	g_{max}
ARMA Filtering(12,12)	0.56 dB	0.68 dB
ARMA Filtering(17,20)	0.31 dB	0.34 dB
AR Filtering(20)	2.6 dB	2.9 dB
AR Filtering(50)	0.26 dB	0.4 dB
IDFT Method $(N = 2^{15})$	0.23 dB	0.29 dB
IDFT Method $(N = 2^{20})$	0.0012 dB	0.0013 dB
SOS (24 Sinusoids)	0.012 dB	0.015 dB

TABLE II

FINITE PRECISION PERFORMANCE COMPARISON AS A FUNCTION OF QUANTIZATION. FOR BOTH MEASURES, 0 DB CORRESPONDS TO PERFECT VARIATE GENERATION.

	Quantization (bits)	$g_{mean}(dB)$	$g_{max}(dB)$
AR(50)	28	.26	.40
AR(50)	24	.28	.41
AR(50)	22	.65	.68
AR(50)	20	fails	
AR(200)	28	.0047	.0048
AR(200)	24	.058	.062
AR(200)	22	fails	
AR(200)	20	fails	
ARMA(12,12)	28	.86	.96
ARMA(12,12)	24	.98	1.04
ARMA(12,12)	22	10.23	10.35
ARMA(12,12)	20	fails	
IDFT(2^{15})	28	.21	.41
IDFT(2^{15})	20	fails	
IDFT(2^{20})	28	.0074	.0076
IDFT(2^{20})	16	3.63	3.68
SOS(24)	28	.012	.015
SOS(24)	10	.011	0.014